Pricing collateralized derivatives with an arbitrary numeraire

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Abstract

Since the 2008 crisis collateralized derivatives have become commonplace in the market. There have been many papers in recent years on pricing collateralized derivatives but the topic has been surrounded by confusion with debate focusing on whether or not a risk-free rate needs to be assumed. In addition, as pointed out by Bielecki and Rutkowski [1], several authors do not pay enough attention to the pricing measure they are working in when setting up their models. The contribution of this paper is to show the pricing formula for a collateralized derivative can be derived under the usual assumptions of an arbitrage-free economy starting from any equivalent martingale measure and associated numeraire.

1 Introduction

Collateralized derivatives have existed in the market for many years. Indeed, futures contracts, which were amongst the earliest of derivatives introduced, are actually examples of collateralized derivatives. But since the financial crisis of 2008 collateralized derivatives have become a lot more important and the focus of a lot of attention.

In Section 2 below we will describe in some generality what a collateralized derivative is and some of the main variants we come across in practice. But throughout this paper we will focus on the standard case, as traded through the various clearing houses now in existence, such as the London Clearing House. We will, for ease of exposition, always talk about the case of a European derivative, but all the results in this paper, with the obvious changes, apply in full generality to path-dependent/American derivatives. We will not consider the issue of counterparties having different funding rates.

Throughout this paper we will consider payments and valuations from the viewpoint of ourselves, one of the counterparties to the transaction being studied. A positive cashflow amount means we will receive the cashflow; a negative cashflow amount means we will pay the cashflow.

For reference, consider first a non-collateralized (European) derivative. At time T this derivative pays the amount V. In order to receive this payment we must pay an amount V_0 today, time zero. So we receive V at T and we receive $-V_0$ at time zero; no other payments are made. It is common to value this product ignoring the credit risk of the transaction, and then to calculate the corrections needed to allow for the credit risk of the two counterparties and the funding cost, the so-called CVA, DVA and FVA.

The key objective behind collateralized derivatives is to minimize credit risk (in an ideal world this would be removed completely, but practicalities mean this cannot be done perfectly, of course). To achieve this, at any time $t \in [0, T]$, the counterparty for which the derivative has negative value must deposit with the other counterparty the value of the derivative. In return, the depositing counterparty receives interest on the collateral posted. Should either counterparty default the trade terminates immediately and the counterparty holding the collateral keeps it (so there is no credit risk).

We will denote by C_t the amount of collateral that we receive at time t. We will use the notation V_t^C to denote the value at time t of the all the payments made if the collateralized derivative is entered at time t.

In order to understand collateralized derivatives we must do two things. First, for a general collateral stream C we must be able to derive an expression for V_t^C. Then the more interesting, and more difficult problem, is to determine for any given derivative payout V and collateral interest-rate μ, what collateral amount will fully remove the credit risk of the trade, i.e., which C will ensure that $V_t^C = C_t$ for all $t \leq T$? We will denote this collateral amount by P and refer to it as the price of the collateralized derivative.

One would typically expect P_t to be close to V_t (the value of the non-collateralized derivative at time t), and usually it is, but it is not equal to V_t (unless the economy is one for which there is a risk-free short-rate and this is the interest rate paid on the collateral). The reason for this is the posting of collateral and the non-standard interest paid on it. For example, if $V > 0$, then $P_t > 0$. If the collateral interest rate is very high, then the counterparty posting collateral will receive an above-market rate of interest on his collateral. This means it is beneficial to post collateral, which means $V_t > P_t$. Put another way, receiving V is less attractive if you have to make lots of high interest payments before you receive it. So a counterparty would be willing to pay less for the collateralized derivative than for the non-collateralized derivative (ignoring CVA, DVA and FVA), $P_t < V_t$.

The challenge of determining P_t given the final 'payout' V and the collateral interest rate that has been agreed $\{\mu_t : 0 \leq t \leq T\}$ turns out, as we shall see, to be quite subtle, and there is not, in general, a unique solution P given the final payout V and the collateral agreement! Fortunately this non-uniqueness does not occur for realistic real-world contracts.

A number of authors have previously considered the problem of collateralized derivatives. But none of the papers so far has fully addressed all the issues. One of the earliest papers, by Piterbarg [11], looked at this problem using replication and the PDE approach. This work was critiqued by Brigio *et al.* [2] who rectified an error in Piterbarg's self-financing condition. However, even with this correction in place the definition of a trading strategy lies outside the standard theory in that it does not comprise only holding assets (see Section 10 for a discussion of how to do this in the standard framework). The pricing of collateralized derivatives using an expectations-based, rather than a PDE-based, approach has also been considered. See, for example the series of papers [4],[5],[6] which are primarily focused on the issue of collateral in a multi-currency setting, and [7] which allows for the possibility of counterparty default (which is relevant when trades are not fully-collateralized).

Before moving on we mention a couple of other papers in the area. In [12] Piterbarg takes a slightly different approach. Recognising the fact that the majority of vanilla interbank derivatve trades are now fully collateralized, Piterbarg proposed a model in which the primitive underlying assets are collateralized derivatives—rather than the standard approach of starting with 'standard' assets then deriving collateralized derivatives. We discuss this in detail in Section 11 where we review the definition of an economy and an equivalent martingale measure (EMM) appropriate for this situation.

Finally, we mention the work of Bielecki and Rutkowski [1]. They consider a much more general situation than we treat here, allowing for funding costs, credit risk and rehypothecation, and they aim to develop a unified martingale framework which sheds light on earlier papers which attempt to incorporate these effects.

All of the approaches so far have worked in the 'risk-neutral' measure with the cash account as numeraire. This raises the question of whether one needs an instantaneous short rate to exist to price a collateralized derivative. The answer is no, it need not exist. We show how to work with a general numeraire (which will always exist). Associated with this numeraire one can find a finite variation process B that, subject to appropriate technical assumptions, can be used to find a tractable expression for the price P. Note that B, although finite variation, may not be absolutely continuous with respect to Lebesgue measure, hence the short-rate may not exist. Furthermore, B may not be a price process, so may not correspond to a cash account and may not have an EMM associated with it.

The layout of this paper, and a summary of our main results, is as follows. In Section 2 we describe in more detail the range of collateralized derivatives that exist in the real world. We introduce the mathematical setup that we use throughout the paper in Section 3, then in Section 4 we consider the (real-world) situation in which collateral is posted at discrete times. (This should not be confused with a discrete-time economy for which prices can only be observed at a discrete set of times. The economy is a continuous-time economy, with asset prices evolving in continuous time. It is just the collateral payments that are made at a set of known discrete times.) We start by considering how to value a derivative with an exogenously specified collateral stream C. We work with a general numeraire N and so we do not need to assume the existence of a short-rate in the economy. Then we use this to derive a simple, explicit expression for the price P^n of a collateralized trade (note that we have here introduced the number of collateral payments n into our notation). The expression we obtain is simple but it is not one that could be used in practice. So we then do some further analysis that results in an implicit characterization of P^n in terms of a martingale $M^{P^n:N}$ associated with the fully-collateralized derivative. This is useful when we move on to the continuous-cashflow case. We finish this section by discussing the 'partial collateral' case for which a proportion $\alpha \neq 1$ of the collateral needed to remove all credit risk is posted.

In Section 5 we introduce a continuous-cashflow model. It is this continuous model that we (and everyone else) analyse in detail. We continue to work with a general numeraire N, rather than a finite-variation one, as is usual. As in the discrete case we first study the case of an exogenous collateral stream C and derive a valuation formula for V^C. We then study the fully-collateralized case, with the collateral stream P, and introduce the martingale $M^{P:N}$ associated with the (numeraire-rebased) cashflows made under the agreement. The martingale $M^{P:N}$ is now central to determining the price P of the fully-collateralized trade. We finish by again discussing the partially-collateralized case, where a proportion α of the full collateral is posted.

Before we can make further progress on pricing in the continuous-cashflow case, we must first identify a finite variation process B associated with the numeraire N. This we do in Section 6. Note that the process B always exists, but need not be the *cash account* which may not exist. But in most cases encountered in practice B will indeed be the cash account. Armed with B we identify a *local* martingale X associated with P. Note that, in general, X need not be a martingale. In the case when X is a martingale, X provides a way to recover P—and in practice X will always be a true martingale. However, in general X may not be a true martingale, and in this case the process P is not uniquely determined. We provide an example to illustrate this.

Armed with B we are now able to examietne more carefully which collateralized derivatives, defined by $(C, \mathcal{M}, V)$ are permitted and can be priced. Of course it is not possible to price everything, even in a complete economy, as we must not allow trading strategies that result in arbitrage. So we need to impose a (standard) L^1 condition on $(C, \mathcal{M}, V)$. Once we have done this we obtain, in Section 7.3, an illuminating formula for the price of the collateralized derivative.

In Section 8, armed with the process B and with the martingale $M^{P:N}$ introduced in Section 5, we return to determining the price of a fully or partially collateralized derivative. We also provide an example to demonstrate that in this setting, unlike in the discrete-cashflow case, the price of a fully-collateralized derivative is not uniquely determined.

Motivated in part by this non-uniqueness, in Section 9 we consider what happens to the discrete-cashflow contract in the limit as the collateral rebalancing frequency increases. We provide a convergence result showing conditions under which the discrete-cashflow contract converges to a continuous-cashflow limit. This is important as it provides conditions under which a continuous-cashflow contract will be a good approximation to the discrete-cashflow real world and shows that when, in the continuous-cashflow setting, the price is not unique we should always use the 'martingale solution'.

In Section 10 we discuss how to obtain a pricing formula for P using a self-financing trading strategy made up of underlying assets and which is then consistent with standard theory, and finally, in Section 11 we discuss Piterbarg's paper [12], *Cooking with Collateral.* By applying the results in this paper we are able to clarify what is done in that paper and how it relates to standard derivatives pricing results. We show that the model Piterbarg introduces is a lot more standard than it might at first appear, and that the measure Piterbarg identifies is indeed just the usual risk-neutral measure.

2 A collateralized derivative: the real-world situation

From Section 5 on, we will consider only the idealized situation in which collateral is posted continuously through time. This is a good approximation to reality and is akin to the near-universal assumption in the derivative-pricing literature that trading happens continuously through time. Here, as there, we move to the continuous setting as the mathematics becomes cleaner and clearer (although moving to a continuous-cashflow stream does introduce some technical issues, as we shall see). But before we take that step we will briefly describe the real-world situation here, and in Section 4 we will discuss in some detail the modelling of the discete-cashflow case.

When two counterparties enter a collateralized derivative, they need to specify the rules by which collateral will be posted. The process of calculating the required collateral and holding it could either be done bilaterally between the two counterparties, or by a clearing house. Banks are steadily being forced by regulators to have clearing houses deal with the collateral. So we will start by describing what happens at the London Clearing House (LCH), one of the main clearing houses.

When two counterparties b and c enter a trade and clear it through the LCH, they must first both post *initial margin* with the clearing house. This is intended to act as a small buffer to allow for the fact that collateral is not posted continuously, and between one posting and the next the derivative will have undergone a discrete change in value. The aim is to ensure, as far as is practically possible, that a counterparty will not default owing money to the LCH, rather the LCH will owe a small amount to the counterparty.

The LCH now starts monitoring the value of this derivative. It runs its servers four times a day, recalculating the value of all trades cleared through it. It will then make a collateral call to all counterparties, or return to them any excess collateral it holds (apart from the small buffer amount mentioned above). Should a counterparty default all positions with that counterparty are closed out as quickly as possible.

Collateral is only accepted in cash in the currency of the original trade (cross-currency trades cannot be cleared through LCH), and interest is paid at the overnight rate for the currency in question—this rate is about as close as one can get, in practice, to a risk-free rate and, as will become clear later, in this case the price of the collateralized trade agrees with the price of the non-collateralized trade (this is also intuitively clear as in this case a counterparty is indifferent to how much collateral it posts, as he gets a market-standard return on the deposit). There are many variants on this approach when counterparties arrange a collateralized trade bilaterally. We will not analyse these variants in this paper, but here we describe them for completeness:

(i) One-way collateral. This is most commonly seen between a bank and a sovereign or a supranational. Governments took the view that they would not default and needed their assets available, so refused to post collateral. But they were concerned banks might default so insisted on banks posting collateral whenever the derivative had positive value for the government. This situation is becoming less common as banks are now not so willing to be exposed to the credit risk of governments, and as banks pay more attention to the actual cost they incur in raising the funds needed to post as collateral.

(ii) Choice over the form and currency of the collateral. Often a collateral agreement will leave some flexibility in the form of the collateral that can be posted. This is useful as a bank may hold bonds it does not want to sell—it can hand these over as security on a trade. It can also be useful for a bank to post collateral in the currency in which it can most easily raise funds. But note that this choice of collateral is in fact an additional embedded option in the trade.

(iii) Minimum transfer amounts. For practical reasons, counterparties may agree only to transfer collateral (in either direction) if the amount owing is large enough. But, of course, this increases the potential loss in the case of a default. Note that this might result in either too much or too little collateral being posted at any given time.

(iv) Thresholds. Sometimes no collateral is posted until the total amount owing is more that a fixed threshold. As with a minimum transfer amount, this increases the credit risk of a transaction.

(v) Posting frequency. The LCH rebalances collateral very frequently, four times a day. But in bilateral agreements this can be done much less often, perhaps daily or weekly.

(vi) Rehypothecation. The analysis in this paper assumes rehypothecation. That is, once a bank receives collateral it can treat it as its own assets. This is important as only then can the receiving bank earn a return on the assets, which is needed for the bank to be able to pay interest on the collateral. Sometimes an agreement might specify that collateral cannot be rehypothecated. In this case the collateral is still useful in mitigating credit risk—the collateral is forfeit on a default. But the collateral holder obtains no funding benefit from holding the collateral, which will affect the value of the collateralized trade.

As mentioned above, all of these variants can and do occur in practice, and they can be valued. But here we only consider the 'standard' LCH case.

3 Mathematical set up

For the purposes of this paper we suppose that our economy consists of a finite number of assets with prices denoted by $A = (A^1, \ldots, A^m)$, where each A^i is modelled as a continuous semimartingale on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ which satisfies the usual conditions.

We make some basic definitions:

Definition 1. *A* generalized price process X *is any* $\{\mathcal{F}_t\}$*-adapted process of the form*

$$X_t := \alpha_t \cdot A_t = X_0 + \int_0^t \alpha_u \cdot dA_u \tag{1}$$

where α is $\{\mathcal{F}_t\}$-predictable. Note that (1) *means the trading strategy α is* self-financing.

Definition 2. *A* numeraire *N is any almost surely strictly positive generalized price process.*

Definition 3. *A measure $\mathbb{N}$ is said to be* an equivalent martingale measure (EMM) *for the economy (corresponding to the numeraire N) if $\mathbb{N} \sim \mathbb{P}$ and A/N is an $\{\mathcal{F}_t\}$ martingale under the measure $\mathbb{N}$.*

Definition 4. *If N is a numeraire and $\mathbb{N}$ is some corresponding EMM we say that $(N, \mathbb{N})$ is a* numeraire pair.

We need to say which trading strategies will be allowed, are *admissible.* We will, of course, insist that our strategies are self-financing (i.e., result in a portfolio value that is a generalized price process.) But, as is well-known, we need to make further restrictions to eliminate so-called 'doubling strategies' from the economy, strategies that introduce arbitrage. There are a number of ways this can be done. We do this in a numeraire-friendly way, by imposing a martingale requirement on the resulting gain process.

Definition 5. *We say that the strategy α is* admissible *with respect to the numeraire pair $(N, \mathbb{N})$ if the resultant portfolio value, $\alpha \cdot A$ is a generalized price process and if $\alpha_t \cdot A_t^N := \alpha_t \cdot A_t / N_t$, is a martingale under the measure $\mathbb{N}$. When this holds we refer to $\alpha \cdot A$ as a* price process.

Note that this definition appears to depend on the numeraire pair $(N, \mathbb{N})$. In fact it is dependent on $\mathbb{N}$ but not on the numeraire N, which is a desirable property. We shall not explore this further here other than through the following definition and result.

Definition 6. *Two numeraire pairs $(N, \mathbb{N})$ and $(\hat{N}, \hat{\mathbb{N}})$ are said to be* equivalent, *$(N, \mathbb{N}) \sim (\hat{N}, \hat{\mathbb{N}})$, if*

$$\left.\frac{d\hat{\mathbb{N}}}{d\mathbb{N}}\right|_{\mathcal{F}_t} := \left(\frac{N_0}{\hat{N}_0}\right) \frac{\hat{N}_t}{N_t}.$$

Lemma 7. *If $(N, \mathbb{N}) \sim (\hat{N}, \hat{\mathbb{N}})$ then the trading strategy α is admissible with respect to the numeraire pair $(N, \mathbb{N})$ if and only if it is admissible with respect to the numeraire pair $(\hat{N}, \hat{\mathbb{N}})$.*

Proof. This follows immediately from [9], Lemma 5.19. □

Lemma 7 shows that admissibility is an equivalence class concept, the ramifications of which are explored further in [10]. Throughout this paper we will work within one equivalence class, and when we say admissible we mean for numeraire pairs in that class. Note that when the economy is complete there is only one equivalence class and the subtleties are not relevant.

Throughout we shall use the notation X^{fv} to denote the finite variation part of any semimartingale X, and X^{loc} to denote the local martingale part. Recall that, given any probability measure $\mathbb{P}$, the decomposition $X - X_0 = X^{\text{loc}} + X^{\text{fv}}$ is unique (although the decomposition would be different under a different measure $\mathbb{Q}$).

4 A discrete cashflow model for the real world

We will now describe mathematically the case when collateral is posted at discrete times which is, of course, always the case in practice. (Note we are working with a continuous-time economy throughout–it is only the payment stream that is discrete.) This allows us to introduce some notation and to draw out some features that are important in the continuous-cashflow-stream setting that follows.

The collateralized trade is just a series of n discrete cashflows, each of which is a function of existing assets in the market. (One could change this assumption and instead take one or more collateralized derivatives as primitives of the economy. This is the viewpoint adopted by Piterbarg [12] (we discuss this approach in Section 11). As such, in principle, it can be valued as standard, with the usual assumptions on no arbitrage and replicability.

Remark 1: Note that in our modelling assumptions we allow for a general numeraire and make no assumption about the existence of a finite variation numeraire for the economy.

Suppose that $V \in m\mathcal{F}_T$ and that the European trade that pays V at time T is a replicable contingent claim for the economy. (Recall that if V is positive we receive money, if V is negative we pay. Other cashflows defined below are interpreted similarly.) Then by the standard theory the value at time t of the (uncollateralized) trade is given by the fundamental valuation formula

$$V_t = N_t \mathbb{E}_{\mathbb{N}} \left[\left. \frac{V}{N_T} \right| \mathcal{F}_t \right]$$

and the process $\{V_t/N_t : 0 \leq t \leq T\}$ is a martingale under $\mathbb{N}$.

For the corresponding collateralized trade, collateral will be posted at the discrete times $0 = t_0^n < t_1^n < t_2^n < \ldots < t_{n-1}^n < T$ and returned, with interest, at the times $0 < t_1^n < t_2^n < \ldots < t_n^n = T$. Note we have introduced the superscript n to denote the total number of collateral payments. This is an unnecessary complication at this point but will prove useful in Section 9 when we discuss convergence.

General collateral

We will denote the collateral payment made at time t by C_t. Recall our convention that $C_t > 0$ means we receive a positive amount. To begin we will consider the case where the amount C_t is some exogenously specified amount–so it will be some $\mathcal{F}_t$-measurable random variable. The question of how to find the amount C_t that fully collateralizes our trade will be considered later.

Under the terms of the collateralized trade, cashflows only occur at the times t_i^n. Suppose we enter the collateralized trade at time t_0 where $t_0 = t_k^n$, for some $k < n$. At time t_0 we receive the amount C_{t_0}. For $t_0 < t_i^n < T$ the net cashflow comprises the current collateral posting $C_{t_i^n}$ plus the return of the previous posting plus interest, $-\left(1 + \mu_{t_{i-1}^n}^n \left(t_i^n - t_{i-1}^n\right)\right) C_{t_{i-1}^n}$. The interest rate μ^n is a rate agreed when the trade is done (there is no need for this rate to be related to a standard market rate, but it usually is). Finally, at time $T = t_n^n$ we receive the previous collateral plus interest, $-\left(1 + \mu_{t_{n-1}^n}^n \left(t_n^n - t_{n-1}^n\right)\right) C_{t_{n-1}^n}$ but make no further collateral posting. Instead we settle the trade, receiving the derivative payoff V.

For convenience we will modify this trade slightly. We will introduce an extra collateral payment C_T made at time T. To offset this, the final derivative settlement amount V will become $V - C_T$. Clearly this modification has no effect–all the net cashflow amounts and cashflow dates are unaltered. It just allows us to simplify our notation in what follows. Note the choice of C_T is arbitrary. However in the continuous-cashflow setting discussed later we want C to be continuous in t so we will take $C_T := \lim_{t \uparrow T} C_t$.

We can now value this trade using the standard arbitrage-pricing approach. For $1 \leq i \leq n$, let $\Gamma_{t_i^n}^n$ denote the collateral-related cashflow at time t_i^n, thus

$$\Gamma_{t_i^n}^n = \left(C_{t_i^n} - C_{t_{i-1}^n}\right) - \mu_{t_{i-1}^n}^n C_{t_{i-1}^n} \left(t_i^n - t_{i-1}^n\right) =: \Delta^n C_{t_{i-1}^n} - \mu_{t_{i-1}^n}^n C_{t_{i-1}^n} \Delta^n t_{i-1}^n.$$

Assuming these cashflows can be replicated (we do not require the economy to be complete) and that the economy is arbitrage-free, then the value of this trade, is given by

$$V^C_{t^n_i} = C_{t^n_i} + N_{t^n_i}\mathbb{E}_{\mathbb{N}}\left[\sum_{j=i+1}^{n}\frac{\Gamma^n_{t^n_j}}{N_{t^n_j}}\middle|\mathcal{F}_{t^n_i}\right] + N_{t^n_i}\mathbb{E}_{\mathbb{N}}\left[\frac{V-C_T}{N_T}\middle|\mathcal{F}_{t^n_i}\right]. \tag{2}$$

Note that V^C depends on the number of payment dates n, so we should really include this in the notation, V^C becoming $V^{n:C}$. But for simplicity we will not make the notation so cumbersome.

Remark 2: Note that in writing (2) we have assumed the usual L^1 restriction on all the cash-flows applies, for example $\mathbb{E}_{\mathbb{N}}\left[|V-C_T|/N_T\right] < \infty$. In the discrete-cashflow setting here this is standard and barely worth highlightly and we will say no more. In the continuous-cashflow case to follow similar L^1 conditions must be enforced. But in that setting the restrictions have a more significant effect.

An important consequence of (2) is the following.

Theorem 8. *Consider a collateralized trade with collateral stream C, final derivative settlement amount V, interest rate on collateral payments μ and where cashflows can only occur at times $0 = t^n_0 < t^n_1 < t^n_2 < \ldots < t^n_n = T$. We assume that the collateral stream C, interest rate μ and derivative payout V are specified so that cashflows can be replicated in our economy and so the value of the trade is given by V^C defined in equation* (2)*.*

Now for $i = 0, \ldots, n$ define

$$V^C_i := V^C_{t^n_i}, \qquad N^n_i := N_{t^n_i} \qquad C^n_i := C_{t^n_i}, \qquad \mathcal{F}^n_i := \mathcal{F}_{t^n_i},$$

and

$$\Phi^n_i := \sum_{j=1}^{i}\frac{\Gamma^n_{t^n_j}}{N_{t^n_j}}. \tag{3}$$

Then

$$M^n_i := \frac{V^C_i - C^n_i}{N^n_i} + \Phi^n_i, \qquad i = 0, \ldots, n, \tag{4}$$

is a discrete-time martingale under $\mathbb{N}$*.*

Proof. The proof follows by observing that equation (2) can be rewritten as

$$\frac{V^C_i - C^n_i}{N^n_i} + \Phi^n_i = \mathbb{E}_{\mathbb{N}}\left[\Phi^n_n + \frac{V-C_T}{N_T}\middle|\mathcal{F}^n_i\right].$$

□

Remark 3: Note that Φ^n in equation (3) has the following interpretation. If at each time $t^n_j, j \le i$, we take the net cashflow $\Gamma^n_{t^n_j}$ arising from the collateral stream and invest it in the numeraire N, then Φ^n_i is the number of units of the numeraire we would hold at time t^n_i. This interpretation of Φ^n as the holding of the numeraire at t^n_i if all income from the collateralized derivative is immediately invested in the numeraire is important. It gives a way, in the continuous-cashflow setting of Section 5, below, to value a collateralized derivative.

Fully-collateralized case

Suppose now we wish to determine the collateral amount required so that either counterparty could exit the trade with zero cost. We will denote this amount P^n. It is chosen so that at time t_i^n+, immediately after the cashflow at time t_i^n, both parties would, in principle, be willing to cancel the trade, along with all future collateral and interest payments, at zero cost. Note that, in order to achieve this, P^n will depend on the interest rate agreed, μ^n, and the collateral posting dates t_i^n, hence the need for the n superscript in the P^n notation. The analysis we, and other authors, have done is about finding the process P^n given the collateral rules.

Remark 4: Were one of the counterparties to default at time t_i^n+, some i, then this collateralization agreement would indeed ensure that neither party suffers a loss due to this default. But, of course, this would never happen in practice. A default would instead occur at a time τ strictly between two collateral payment dates, $t_{i-1}^n < \tau < t_i^n$. This means there could be a credit loss, but if the collateral dates are close together, this loss would be small as the interest due will be small and the market will not have moved much over the time period $[t_{i-1}^n, \tau]$.

Remark 5: In this paper, we will only need to consider P^n at dates $t_i^n, i = 0, \ldots, n$. However it is natural to extend its definition to arbitrary times. For $t \notin \{t_n^i : i = 0, \ldots, n\}$, P_t^n is the amount of collateral to be posted at time t were the trade to be entered at time t. Note however that in the fully-collateralized case there will be no net cashflow at t ($P_t^n = V_t^{P^n}$), the time we enter the trade-collateral package: we pay P_t^n for the collateralized trade and P_t^n of collateral is posted by our counterparty—resulting in zero net cashflow at t.

Remark 6: Note that $P_{t_n^n}^n = P_T^n = V$ (there will be no interest payments if the trade is entered at T and the trade will settle immediately, paying V).

Theorem 9. *Consider a collateralized trade as described in Theorem 8 but where the collateral stream, now denoted by P^n, is to be specified such that the value of the trade at times $t_i^n, i = 1, \ldots, n$ is equal to the collateral posted at that time and so either counterparty can exit the trade at zero cost. In this discrete-cashflow setting we have the following explicit expression for P^n*

$$P_{t_i^n}^n = \frac{\mathcal{M}_{t_i^n}^n N_{t_i^n}}{B_{t_i^n}^n} \mathbb{E}_{\mathbb{N}} \left[\left. \frac{V B_T^n}{\mathcal{M}_T^n N_T} \right| \mathcal{F}_{t_i^n} \right], \tag{5}$$

where

$$\mathcal{M}_{t_i^n}^n := \prod_{j=0}^{i-1} \left(1 + \mu_{t_j^n}^n \Delta^n t_j^n\right), \qquad B_{t_i^n}^n := \prod_{j=0}^{i-1} \left(1 + r_{t_j^n}^n \Delta^n t_j^n\right),$$

and

$$r_{t_i^n}^n := \frac{1 - D_{t_i^n t_{i+1}^n}}{D_{t_i^n t_{i+1}^n} \Delta^n t_i^n}, \qquad \left(\text{i.e., } 1 + r_{t_i^n}^n \Delta^n t_i^n = D_{t_i^n t_{i+1}^n}^{-1}\right).$$

Here $D_{t_i^n t_{i+1}^n}$ is a standard discount factor

$$D_{t_i^n t_{i+1}^n} := N_{t_i^n} \mathbb{E}_{\mathbb{N}} \left[\left. \frac{1}{N_{t_{i+1}^n}} \right| \mathcal{F}_i^n \right].$$

Proof. We prove this by using induction, working backwards through the times t_i^n, starting with $t_n^n = T$.

The result is clearly true at time $t_n^n = T$. So suppose now that (5) holds for $j > i$. Note first that in this case the discrete-time martingale M^n in (4) is just Φ^n since $V_i^{P^n} = P_i^n$ (the collateral posted is precisely the value of the collateralized trade), hence

$$\begin{aligned} 0 &= \mathbb{E}_{\mathbb{N}}\left[M_{i+1}^n - M_i^n \middle| \mathcal{F}_i^n\right] = \mathbb{E}_{\mathbb{N}}\left[\Phi_{i+1}^n - \Phi_i^n \middle| \mathcal{F}_i^n\right] \\ &= \mathbb{E}_{\mathbb{N}}\left[\frac{\Gamma_{t_{i+1}^n}^n}{N_{t_{i+1}^n}}\middle| \mathcal{F}_i^n\right] = \mathbb{E}_{\mathbb{N}}\left[\frac{P_{t_{i+1}^n}^n - P_{t_i^n}^n\left(1+\mu_{t_i^n}^n\Delta^n t_i^n\right)}{N_{t_{i+1}^n}}\middle| \mathcal{F}_i^n\right] \\ &= \mathbb{E}_{\mathbb{N}}\left[\frac{P_{t_{i+1}^n}^n}{N_{t_{i+1}^n}}\middle| \mathcal{F}_i^n\right] - \frac{P_{t_i^n}^n}{N_{t_i^n}}\frac{1+\mu_{t_i^n}^n\Delta^n t_i^n}{1+r_{t_i^n}^n\Delta^n t_i^n}. \end{aligned} \tag{6}$$

Thus

$$\frac{P_{t_i^n}^n}{N_{t_i^n}}\frac{B_{t_i^n}^n}{\mathcal{M}_{t_i^n}^n} = \mathbb{E}_{\mathbb{N}}\left[\frac{P_{t_{i+1}^n}^n}{N_{t_{i+1}^n}}\frac{B_{t_{i+1}^n}^n}{\mathcal{M}_{t_{i+1}^n}^n}\middle| \mathcal{F}_i^n\right].$$

Finally we apply the inductive hypothesis

$$\frac{P_{t_i^n}^n}{N_{t_i^n}}\frac{B_{t_i^n}^n}{\mathcal{M}_{t_i^n}^n} = \mathbb{E}_{\mathbb{N}}\left[\mathbb{E}_{\mathbb{N}}\left[\frac{VB_T^n}{\mathcal{M}_T^n N_T}\middle| \mathcal{F}_{i+1}^n\right]\middle| \mathcal{F}_i^n\right] = \mathbb{E}_{\mathbb{N}}\left[\frac{VB_T^n}{\mathcal{M}_T^n N_T}\middle| \mathcal{F}_i^n\right],$$

as required. □

Remark 7: Equation (5) shows that $P^n B^n/\mathcal{M}^n N$ is a martingale in the measure $\mathbb{N}$, and (5) completely characterises P^n. But both B^n and $\mathcal{M}^n$ are path dependent and so difficult to work with. In the next section we will switch to the continuous-cashflow setting.

Remark 8: Equation (6) is just the statement that at time t_i^n the cashflow due at t_{i+1}^n has value zero. Similarly we can conclude that at time t_i^n the cashflow due at t_j^n has value zero for any $j > i$.

Remark 9: In the continuous-cashflow setting some things simplify. On the other hand, in some ways continuous cashflows make things harder. It is no longer possible to work backwards one step at a time (there are no discrete steps any more). As a result we need to derive P in a different way, and the uniqueness of the solution P is lost. This is because, in the continuous case, $PB/\mathcal{M}N$ is only a local martingale and not a true martingale in general (in discrete time all local martingales are in fact true martingales—see Protter [13], Theorem 47, for a precise statement of what is true).

5 A continuous-cashflow model for collateralization approximating the real world

We now begin our study of the case where we assume collateral is posted continuously rather than at discrete times. In Section 9 we will comment on in what sense this idealized framework can be viewed as a limit of the discrete-cashflow setup.

By moving to the continuous-cashflow case we can obtain explicit and tractable results for the price P of a fully (or partially) collateralized trade. These results will be developed in Section 8. Here we will develop an expression for the analogue of the martingale M^n in the continuous-cashflow case.

As in the discrete-cashflow case we will first examine the case of an exogenously specified collateral stream. Let C denote the collateral process. Further let μ_t denote the rate of interest at time t as agreed by the two counterparties when entering the trade. In practice this collateral rate may be related to some market rate but from a technical point of view we need only assume that the process μ is adapted to the filtration of our economy and is such that various expectations below are defined. Note the collateral rate can be deal specific.

Let V_t^C denote the value of the trade if it is entered at time t. This trade comprises the original collateral posting C_t, paid at time t, the final settlement amount V paid at T, plus the value of the cashflows arising from the income stream generated from the rebalacing of the collateral process C and the interest paid thereon. As in the discrete-cashflow case above it will be convenient notationally to introduce an extra collateral payment $C_T := \lim_{t\uparrow T} C_t$ at T and to compensate for this by changing the final settlement amount from V to $V - C_T$. Note this has no net effect.

Cashflow valuation at t: Clearly the original collateral posting at t is worth C_t. With the usual assumptions of no arbitrage and replicability, the final settlement is worth

$$N_t \mathbb{E}_{\mathbb{N}} \left[\frac{V - C_T}{N_T} \middle| \mathcal{F}_t \right].$$

It remains to value the continuous collateral stream. This is done in the following theroem.

Theorem 10. *Let V_t^C denote the value of a collateralized trade entered at time t having continuous collateral stream C, final derivative settlement amount V and interest rate on collateral payments μ. We assume that the collateral stream C and collateral interest rate μ are such that investing all income from the collateral stream in the numeraire N is an admissible trading strategy.*

Set

$$\mathcal{M}_t := \exp\left(\int_0^t \mu_s ds \right)$$

and define

$$\Phi_t^{C:N} := \int_0^t \frac{\mathcal{M}_s}{N_s} \left(d\left(\frac{C_s}{\mathcal{M}_s} \right) - \frac{1}{N_s} d\left[\frac{C}{\mathcal{M}}, N \right]_s \right).$$

We suppose that the collateralized derivative can be replicated by an admissible trading strategy. Then

$$\frac{V_t^C - C_t}{N_t} + \Phi_t^{C:N} = \mathbb{E}_{\mathbb{N}} \left[\frac{V - C_T}{N_T} + \Phi_T^{C:N} \middle| \mathcal{F}_t \right], \tag{7}$$

and thus

$$M_t^{C:N} := \frac{V_t^C - C_t}{N_t} + \Phi_t^{C:N} \tag{8}$$

is a martingale under $\mathbb{N}$.

Proof. Let I_t^C denote the total income received by t from the collateral stream, which we know is given by

$$I_t^C := \int_0^t dC_s - \mu_s C_s ds = \int_0^t \mathcal{M}_s d\left(\frac{C_s}{\mathcal{M}_s} \right). \tag{9}$$

Now let $\Phi_t^{C:N}$ denote the holding at t of the numeraire if the trade is entered at time zero and all proceeds from the continuous collateral stream are invested in the numeraire N. The value, at t, of our numeraire holding is just $\Phi_t^{C:N} N_t$. The self-financing condition applied to this holding requires

$$
\begin{aligned}
dI_t^C + \Phi_t^{C:N} dN_t &= d\left(\Phi_t^{C:N} N_t\right) = \Phi_t^{C:N} dN_t + N_t d\Phi_t^{C:N} + d\Phi_t^{C:N} dN_t, \\
&= \Phi_t^{C:N} dN_t + N_t d\Phi_t^{C:N} + d[\Phi^{C:N}, N]_t,
\end{aligned}
$$

hence

$$
\begin{aligned}
dI_t^C &= N_t d\Phi_t^{C:N} + d[\Phi^{C:N}, N]_t, \\
d\Phi_t^{C:N} &= \frac{dI_t^C}{N_t} - \frac{d[\Phi^{C:N}, N]_t}{N_t} = \frac{dI_t^C}{N_t} - \frac{d[I^C, N]_t}{N_t^2}.
\end{aligned}
$$

Thus, substituting (9),

$$
\Phi_t^{C:N} = \int_0^t \frac{dI_s^C}{N_s} - \frac{d[I^C, N]_s}{N_s^2} = \int_0^t \frac{\mathcal{M}_s}{N_s} \left(d\left(\frac{C_s}{\mathcal{M}_s}\right) - \frac{1}{N_s} d\left[\frac{C}{\mathcal{M}}, N\right]_s \right).
$$

Suppose now we entered the trade at time t. To do so we must pay $V_t^C - C_t$ (we pay for the trade and receive collateral). If all proceeds from the collateral stream are invested in the numeraire then at T we will hold $\Phi_T^{C:N} - \Phi_t^{C:N}$ units of the numeraire. If we now liquidate our holding at time T our wealth will be

$$
V - C_T + N_T \left(\Phi_T^{C:N} - \Phi_t^{C:N}\right).
$$

Assuming this can be replicated by an admissible strategy in an arbitrage-free economy, we can value this payoff in the usual way, and this value will be the amount we paid when entering the trade:

$$
V_t^C - C_t = N_t \mathbb{E}_{\mathbb{N}} \left[\frac{V - C_T}{N_T} + \Phi_T^{C:N} - \Phi_t^{C:N} \middle| \mathcal{F}_t \right].
$$

This is precisely equation (7). □

Putting this all together, assuming that this collateralized derivative can be replicated via some admissible trading strategy, we obtain

$$
V_t^C = C_t + N_t \mathbb{E}_{\mathbb{N}} \left[\int_0^t \frac{\mathcal{M}_s}{N_s} \left(d\left(\frac{C_s}{\mathcal{M}_s}\right) - \frac{1}{N_s} d\left[\frac{C}{\mathcal{M}}, N\right]_s \right) \middle| \mathcal{F}_t \right] + N_t \mathbb{E}_{\mathbb{N}} \left[\frac{V - C_T}{N_T} \middle| \mathcal{F}_t \right].
$$

The second term is the value of the continuous collateral stream.

Remark 10: In the statement of Theorem 10 we have assumed that investing the cashflows from the collateral stream in the numeraire N is an admissible trading strategy. This is not true for all C, N and $\mathcal{M}$ - some cases will result in a numeraire-rebased gain process that is a strict local martingale, something that can result in arbitrage (for example via a 'doubling-type' strategy) and needs to be excluded. We discuss this in Section 7 after we have introduced the finite variation process B in Section 6.

Remark 11: In our discussions so far we have talked about the interest rate μ paid on collateral. We then defined the 'collateral account' $\mathcal{M}$ which is of finite variation, and indeed absolutely continuous with respect to Lebesgue measure. In fact, for all the results in this paper to hold we only need $\mathcal{M}$ to be of finite variation. Absolute continuity is not needed.

Fully-collateralized case

Suppose we have entered a trade in which we receive V at time T and a full collateral agreement, i.e., the collateral P_t posted at any time t is exactly the amount needed to ensure that either counterparty would be willing to cancel the trade at zero cost.

In this case the $V_t^P = P_t$, all t, and so $M^{P:N} = \Phi^{P:N}$ and $\Phi^{P:N}$ is a martingale under $\mathbb{N}$. This observation is key to finding the valuation formula for the fully-collateralized trade but first we must prove a few technical lemmas.

Recalling that

$$\Phi_t^{P:N} := \int_0^t \frac{\mathcal{M}_s}{N_s} \left(d\left(\frac{P_s}{\mathcal{M}_s} \right) - \frac{1}{N_s} d\left[\frac{P}{\mathcal{M}}, N \right]_s \right),$$

it is clear that the martingale property of $\Phi^{P:N}$ only implicitly defines the process P. In fact it does not uniquely characterize P. In Section 8 we give an example of a fully-collateralized trade in which P is not uniquely determined.

We return to solving for P in Section 8.

Partially-collateralized case

An important case in practice is that of partial collateral in which $C_t = \alpha_t V_t^C$ for some $\{\mathcal{F}_t\}$-adapted process α. For example collateral may only be posted when it exceeds some minimum threshold.

Note that we could have expanded our treatment of the discrete-cashflow case to include partial collateral. Given the discussion here that extension is straightforward so we have omitted it for brevity.

In general this is a hard case to solve because α depends on V^C which in turn depends on α. But theoretically, at least, it can be reduced to the fully-collateralized case above. We will discuss this further in Section 8.2 after we have introduced the finite variation process B.

Remark 12: We refer to this as partial collateral, but a better name would be *imperfect collateral* as sometimes $\alpha_t > 1$. For example, this could occur if collateral is only posted/returned in discrete amounts (multiples of $1 million).

In the partially-collateralized case in which we receive $\alpha_T V$ at time T let P_t^α denote the value of the collateral payment at time t, $\alpha_t P_t^\alpha$, plus the value of all future cashflows arising from the income stream generated from the collateral process αP^α. Here the net cashflow at time t will be $\alpha_t P_t^\alpha - P_t^\alpha$ and, it follows from the discussion above that the collateral stream P^α must be chosen so that

$$\frac{(1-\alpha)P^\alpha}{N} + M^{\alpha P^\alpha : N}$$

is a martingale under $\mathbb{N}$.

We will see in Section 8 that the partially collateralized case can be reduced to the fully collateralized one by considering an alternative collateral interest rate. With this alternative collateral interest rate P^α can be viewed as the price of a fully-collateralized derivative and so the valuation formula for the fully-collateralized case will apply.

6 The finite variation process B

By considering the collateral stream in Section 5 we obtained a martingale (the process $M^{C:N}$ defined in equation (8)) that involved the numeraire, N, the process describing the agreed interest on collateral, $\mathcal{M}$, and the price of the collateralized derivative itself, V^C. For a fully-collateralized derivative, in the case where the numeraire N is of finite variation the martingale property of $M^{P:N}$ implies that $P/\mathcal{M}$ is a (local) martingale, whence (under appropriate boundedness conditions) we obtain a formula for the price of the fully-collateralized trade at time t, P_t.

If the numeraire N is not of finite variation there is some more to do. We first need to construct a finite variation process, B, which provides the bridge to the final pricing formula. We shall do this now in Lemma 11 below and in this section we will also discuss properties of B (Is it unique? Is it the cash account?). Then, in Section 8.1 we will use this process to recover an expression for the fully-collateralized derivative's price P. As we shall see, in the continuous-cashflow setting here things are not as straightforward as in the discrete-cashflow setting (as not every local martingale is a true martingale). We include an example to show how things can get more complicated.

Lemma 11. *Given a numeraire pair $(N, \mathbb{N})$, where recall N is a positive continuous semi-martingale, define the process B via*

$$B_t := \exp\left(\int_0^t \frac{dN_s^{\text{fv}}}{N_s} - \frac{d[N]_s}{N_s^2}\right),$$

where N^{fv} denotes the finite variation part of the numeraire N under $\mathbb{N}$. This process B is the unique (strictly positive) finite variation process with $B_0 = 1$ such that B/N is a local martingale with respect to $(\{\mathcal{F}_t\}, \mathbb{N})$.

Proof. Clearly the process B defined above is of finite variation being the exponential of the sum of integrals against finite variation processes, and $B_0 = 1$. Applying Itô's formula to B/N we see that B/N is indeed a local martingale, as claimed.

To prove that B is the unique process with these properties, suppose $\hat{B}$ is of finite variation, $\hat{B}_0 = 1$, and $\hat{B}/N$ is a local martingale. Then, by Itô's formula,

$$\begin{aligned} d\left(\frac{\hat{B}_t}{N_t}\right) &= \hat{B}_t dN_t^{-1} + N_t^{-1} d\hat{B}_t + d\hat{B}_t dN_t^{-1} = \hat{B}_t\left(-\frac{dN_t}{N_t^2} + \frac{d[N]_t}{N_t^3}\right) + \frac{d\hat{B}_t}{N_t} \\ &= -\frac{\hat{B}_t}{N_t}\frac{dN_t^{\text{loc}}}{N_t} + \left\{\hat{B}_t\left(-\frac{dN_t^{\text{fv}}}{N_t^2} + \frac{d[N]_t}{N_t^3}\right) + \frac{d\hat{B}_t}{N_t}\right\}. \end{aligned} \tag{10}$$

For $\hat{B}/N$ to be a local martingale, the final (finite variation) term in curly brackets in equation (10) must be zero. That is,

$$d\hat{B}_t = \hat{B}_t dZ_t, \tag{11}$$

where

$$dZ_t := \frac{dN^{\text{fv}}}{N_t} - \frac{d[N]_t}{N_t^2}.$$

Equation (11) for $\hat{B}$, being the Doléans exponential of the finite variation process Z, has a unique solution, namely

$$\hat{B}_t = \hat{B}_0 \exp\left(Z_t - Z_0 - \tfrac{1}{2}[Z]_t\right) = \exp\left(\int_0^t \frac{dN_s^{\text{fv}}}{N_s} - \frac{d[N]_s}{N_s^2}\right) = B_t.$$

That is, $\hat{B} = B$ and so B is indeed the unique process with the stated properties. □

The process B introduced in Lemma 11 is defined in terms of a numeraire N and a martingale measure $\mathbb{N}$. This raises the question of whether B really does depend on $(N, \mathbb{N})$ or whether using a different numeraire pair $(\hat{N}, \hat{\mathbb{N}})$ would yield the same process $\hat{B}$. (Answering this question is not necessary for the purposes of this paper, but it is an obvious question to ask.) The following three results provide an answer.

Corollary 12. *If $(N, \mathbb{N})$ and $(\hat{N}, \hat{\mathbb{N}})$ are two equivalent numeraire pairs, then*

$$B_t := \exp\left(\int_0^t \frac{dN_s^{\text{fv}}}{N_s} - \frac{d[N]_s}{N_s^2}\right) = \exp\left(\int_0^t \frac{d\hat{N}_s^{\text{fv}}}{\hat{N}_s} - \frac{d[\hat{N}]_s}{\hat{N}_s^2}\right) =: \hat{B}_t.$$

Proof. This follows immediately from the uniqueness of B/N in Lemma 11. □

As a consequence of Corollary 12, B is unique if the economy is complete.

Lemma 13. *If the process B defined in Lemma 11 is a generalized price process (recall Definition 1) then the process B is independent of the numeraire pair used to define it.*

Proof. Suppose that B is the process arising from the numeraire pair $(N, \mathbb{N})$ via Lemma 11 and that B is also a generalized price process, meaning

$$B_t = B_0 + \int_0^t \sum_{i=1}^m \varphi_s^i dA_s^i = \sum_{i=1}^m \varphi_t^i A_t^i,$$

for some predictable process φ. Suppose $\hat{B}$ is some other finite-variation process derived from another numeraire pair, $(\hat{N}, \hat{\mathbb{N}})$. Noting that the sequence of stopping times $\{T_k\}$ defined for each $k \in \mathbb{Z}^+$ by

$$T_k := \inf\left\{t > 0 : \left|\hat{B}_t/\hat{N}_t\right| > k\right\}$$

is a reducing sequence for the local martingale $\hat{B}/\hat{N}$ we can define a measure $\hat{\mathbb{B}}^k$ equivalent to $\hat{\mathbb{N}}$ via

$$\left.\frac{d\hat{\mathbb{B}}^k}{d\hat{\mathbb{N}}}\right|_{\mathcal{F}_t} := N_0 \left(\frac{\hat{B}}{\hat{N}}\right)_t^{T_k}.$$

Since $(\hat{N}, \hat{\mathbb{N}})$ is a numeraire pair for the economy, the stopped process $\left(A/\hat{N}\right)^{T_k}$ is a martingale under $\hat{\mathbb{N}}$ and by a standard result (see for example [9], Lemma 5.19) we have that

$$\left(\frac{A}{\hat{N}}\right)^{T_k} \left.\frac{d\hat{\mathbb{N}}}{d\hat{\mathbb{B}}^k}\right|_{\mathcal{F}_t} = \left(\frac{A}{\hat{B}}\right)^{T_k}$$

is a martingale under $\hat{\mathbb{B}}^k$. By unit invariance (which follows immediately from Itô's formula) we have

$$\frac{B_t}{\hat{B}_t} = \frac{B_0}{\hat{B}_0} + \int_0^t \sum_{i=1}^m \varphi_s^i d\left(\frac{A^i}{\hat{B}}\right)_s.$$

Since stopping a stochastic integral is equivalent to stopping the integrator the stopped process $\left(B/\hat{B}\right)^{T_k}$ is a local martingale under $\hat{\mathbb{B}}^k$. It is also a continuous finite variation process, being the ratio of two continuous finite variation processes. So it must be constant, that constant being one as $B_0 = \hat{B}_0 = 1$. Letting $k \to \infty$ we can conclude $B_t = \hat{B}_t$, almost surely, for all t. $\square$

Lemma 14. *In general (in an incomplete economy) the process B defined in Lemma 11 depends on the numeraire pair $(N, \mathbb{N})$ used to define it.*

Proof. We prove this result by describing how two such distinct processes B and $\hat{B}$ can be constructed using different numeraire pairs.

Consider an economy $\mathcal{E}$ for which there exists more than one EMM corresponding to numeraire N, and denote two of these measures by $\mathbb{N}$ and $\hat{\mathbb{N}}$. This will be the case if the economy is incomplete.

Let B denote the unique finite variation process such that B/N is a local martingale under $\mathbb{N}$. By Lemma 11, B is given by

$$B_t = \exp\left(\int_0^t \frac{dN_s^{\text{fv}}}{N_s} - \frac{d[N]_s}{N_s^2}\right), \tag{12}$$

where N^{fv} denotes the finite variation part of the numeraire N under $\mathbb{N}$. We now derive an expression for $\hat{B}$, defined using the numeraire pair $(N, \hat{\mathbb{N}})$, which is different from (12), and thus we establish that $B \neq \hat{B}$.

Now let $\hat{B}$ denote the unique finite variation process such that $\hat{B}/N$ is a local martingale under $\hat{\mathbb{N}}$, and define

$$\rho_t := \left.\frac{d\hat{\mathbb{N}}}{d\mathbb{N}}\right|_{\mathcal{F}_t},$$

the Radon-Nikodym derivative connecting $\mathbb{N}$ and $\hat{\mathbb{N}}$. It is a standard result (see for example Lemma 5.19 of [9]), that M is a local martingale under $\hat{\mathbb{N}}$ if and only if ρM is a local martingale under $\mathbb{N}$. Thus $\rho(\hat{B}/N)$ is a local martingale under $\mathbb{N}$.

Applying Itô's formula to $\rho(\hat{B}/N)$ yields

$$d\left\{\rho_t\left(\frac{\hat{B}_t}{N_t}\right)\right\} = \left\{\rho_t d\left(\frac{\hat{B}_t}{N_t}\right)^{\text{fv}} + d\rho_t d\left(\frac{\hat{B}_t}{N_t}\right)\right\} + \left\{\rho_t d\left(\frac{\hat{B}_t}{N_t}\right)^{\text{loc}} + \left(\frac{\hat{B}_t}{N_t}\right) d\rho_t\right\}. \tag{13}$$

Since $\rho(\hat{B}/N)$ is a local martingale, the finite variation part of (13) is zero:

$$\rho_t d\left(\frac{\hat{B}_t}{N_t}\right)^{\text{fv}} + d\rho_t d\left(\frac{\hat{B}_t}{N_t}\right) = 0 \tag{14}$$

where here $\left(\hat{B}/N\right)^{\text{fv}}$ denotes the finite variation part of $\left(\hat{B}/N\right)$ under $\mathbb{N}$. Noting $\hat{B}$ will still be a finite varaiation process under $\mathbb{N}$, we can expand out the term $d\left(\hat{B}_t/N_t\right)$ in (14) to obtain

$$\rho_t\left(\frac{1}{N_t} d\hat{B}_t - \frac{\hat{B}_t}{N_t^2} dN_t^{\text{fv}} + \frac{\hat{B}_t}{N_t^3} d[N]_t\right) - \frac{\hat{B}_t}{N_t^2} d\rho_t dN_t^{\text{loc}} = 0,$$

thus

$$d\hat{B}_t = \hat{B}_t \left(\frac{dN_t^{\text{fv}}}{N_t} - \frac{d[N]_t}{N_t^2} + \frac{d\rho_t dN_t^{\text{loc}}}{\rho_t N_t} \right). \tag{15}$$

Equation (15) is a Doléans exponential with unique solution (with $\hat{B}_0 = 1$) given by

$$\hat{B}_t = \exp\left(\int_0^t \frac{dN_s^{\text{fv}}}{N_s} - \frac{d[N]_s}{N_s^2} + \frac{d\rho_t d{N_t}^{\text{loc}}}{\rho_t N_t} \right) = B_t \exp\left(\int_0^t \frac{d\rho_t dN_t}{\rho_t N_t} \right).$$

The processes B and $\hat{B}$ will differ as long as $d\rho_t dN_t \neq 0$. $\square$

Example 1: The proof above demonstrates how any two processes B and $\hat{B}$ with the required properties are related to each other. Now we provide an explicit example of an (incomplete) economy for which uniqueness of B does not hold. Let (W_1, W_2) be a standard Brownian motion with $dW_1 dW_2 = 0$. Define two positive assets (numeraires) N_1 and N_2 by

$$\frac{dN_i(t)}{N_i(t)} = \mu_i(t)dt + \sigma_i(t)dW_i(t), \tag{16}$$

for $i = 1, 2$, where μ_i and σ_i are chosen so that the SDE's (16) have a strong solution. For any suitable predictable process η we can change measure so that $(\tilde{W}_1, \tilde{W}_2)$ is a standard Brownian motion, where

$$\begin{aligned} d\tilde{W}_1(t) &:= \frac{\mu_1(t) - \eta(t) + \sigma_2^2(t)}{\sigma_1(t)} dt + dW_1(t), \\ d\tilde{W}_2(t) &:= \frac{\mu_2(t) - \eta(t)}{\sigma_2(t)} dt + dW_2(t). \end{aligned}$$

Substituting into (16) yields

$$\begin{aligned} \frac{dN_1(t)}{N_1(t)} &= \left(\eta(t) - \sigma_2^2(t)\right) dt + \sigma_1(t) d\tilde{W}_1(t), \\ \frac{dN_2(t)}{N_2(t)} &= \eta(t)dt + \sigma_2(t) d\tilde{W}_2(t), \end{aligned}$$

whence

$$d\left(\frac{N_1}{N_2}\right)_t = \left(\frac{N_1}{N_2}\right)_t \left(\sigma_2(t) d\tilde{W}_2(t) + \sigma_1(t) d\tilde{W}_1(t)\right).$$

Thus N_1/N_2 is a martingale under this measure $\mathbb{N}_2$, and $(N_2, \mathbb{N}_2)$ is a numeraire pair. The corresponding finite-variation process B is given by

$$B_t = \exp\left(\int_0^t \frac{dN_2^{\text{fv}}(s)}{N_2(s)} - \frac{d[N_2](s)}{N_2^2(s)} \right) = \exp\left(\int_0^t \left(\eta(s) - \sigma_2^2(s)\right) ds \right).$$

Recalling that η was an arbitrary choice we see that the positive finite variation process B is not unique.

Remark 13: Even if the process B satisfies the conditions of Lemma 13 it is not necessarily a price process (so in particular B/N is not necessarily a martingale) and there may not exist an equivalent martingale measure corresponding to B taken as numeraire. This is true even if the economy is complete– see Theorem 7.43 of [9].

If B is a price process and so B/N is a martingale then we can construct an EMM $\mathbb{B}$ corresponding to B taken as numeraire via

$$\left.\frac{d\mathbb{B}}{d\mathbb{N}}\right|_{\mathcal{F}_t} := N_0 \frac{B_t}{N_t}.$$

Even if B is a price process, under the general assumptions on our economy it may be the case that $\log B$ is not absolutely continuous with respect to Lebesgue measure and so a risk-free rate may not exist for the economy. Only in the case when $\log B$ is absolutely continuous with respect to Lebesgue measure can we write B in the form

$$B_t = \exp\left(\int_0^t r_s ds\right). \tag{17}$$

But even when the measure $\mathbb{B}$ exists and when we can write (17), it does not imply that r has the interpretation of being a short rate in the sense that $r_t = -\lim_{h\to 0} \log D_{t,t+h}/h$ where, for $S > t, D_{tS}$ is the value at t of a bond that matures worth one at time S, i.e., $D_{tS} = B_t \mathbb{E}_{\mathbb{B}}\left[B_S^{-1}\middle|\,\mathcal{F}_t\right]$. The existence of a risk-free rate has been the starting point for several authors and lack of clarity over its role has been evident in several accounts (see, for example, [7] and references therein).

7 Admissibility and an alternative formula for the value of a collateralized trade

Throughout we have invested all income arising from the collateralized derivative in the numeraire. But doing so is not an admissible trading strategy for all combinations of C and N. We commented on this in Section 5 but did not discuss it further then as this is more easily done now that we have met the process B. We shall discuss this shortly in Section 7.1

We will not be able to price all collateralized derivatives, i.e., for all triples $(C, \mathcal{M}, V)$. We alluded to this earlier but only now that we have met B can we state the restrictions. We shall impose an L^1 condition on the contract, which we do in Section 7.2.

Once we have imposed both these restrictions we are able to derive an explicit formula for V^C, the value of the collateralized derivative. We do this and explore some of its ramifications in Section 7.3.

7.1 Reinvestment restrictions

Not all trading strategies are admissible. We must eliminate all ‘doubling strategies’ that can result in arbitrage and we do this by insisting that the numeraire-rebased gain process of a self-financing trading strategy must be a martingale under the corresponding EMM (clearly this is always a local martingale; we insist it isn’t a strict local martingale).

In the context of a collateralized derivative this results in the following. Define

$$R_t^{C:N} := \int_0^t \frac{C_s}{N_s^2} dN_s^{\mathrm{loc}}. \tag{18}$$

Clearly $R^{C:N}$ is a local martingale (under the measure $\mathbb{N}$).

- We insist that $R^{C:N}$ is a martingale under the measure $\mathbb{N}$.

This is a natural restriction as $R^{C:N}$ is the numeraire-rebased gain process corresponding to trading strategy, as we now show. Requiring that it is a martingale is thus just our usual admissibility requirement for a trading strategy. If $R^{C:N}$ is a strict local martingale we do not allow the collateral stream to be invested in N.

Suppose for now that B is a price process, either an asset in the economy or something that could be produced by trading in other assets. We do not need this assumption and will remove it shortly. But it is easier to understand our trading strategies if we start with the assumption.

Suppose we start with zero wealth at time zero. At any time $t \in [0, T]$ we hold α_t units of B and β_t units of the numeraire. We take $\alpha_t := -C_t/B_t$ and invest the rest of our wealth in the numeraire. This results in a numeraire holding

$$\beta_t := \hat{\Phi}_t^{C:N} := \int_0^t \frac{B_s}{N_s}\left(d\left(\frac{C_s}{B_s}\right) - \frac{1}{N_s} d\left[\frac{C}{B}, N\right]_s\right).$$

It is straightforward to check that the numeraire-rebased gain process arising from this self-financing strategy is precisely $R^{C:N}$,

$$R_t^{C:N} = \frac{\alpha_t B_t + \beta_t N_t}{N_t}.$$

We thus insist $R^{C:N}$ is a martingale to ensure this strategy is not an arbitrage.

Suppose now that we generalize this strategy. We combine this strategy using the numeraire N^1 with the opposite strategy using a second numeraire N^2. That is, we hold $\beta_t^1 = \hat{\Phi}_t^{C:N^1}$ units of N^1, $\beta_t^2 = -\hat{\Phi}_t^{C:N^2}$ units of N^2, and $\alpha_t = -C_t/B_t + C_t/B_t = 0$ units of B. So we don't hold B and it does not need to be a tradable asset in the economy.

The gain from this strategy is

$$G_t = N_t^1 R_t^{C:N^1} + N_t^2 R_t^{C:N^2},$$

which, when numeraire-rebased, must be a martingale to exclude arbitrage. If N^1 and N^2 are price process with respect to the numeraire pair $(N, \mathbb{N})$ this follows from the martingale property of each $R^{C:N^i}$ in its respective martingale measure.

Remark 14: In general, for any particular choice of numeraire pair $(N, \mathbb{N})$ there is no guarantee that $R^{C:N}$ will be a martingale but we would expect there is at least one numeraire for which this holds. In particular, this will be true if B is an asset in the economy-in this case $R^{C:B} = 0$, a martingale.

Remark 15: Using the process B, $R^{C:N}$ can be written more simply as

$$R_t^{C:N} := \int_0^t \frac{C_s}{N_s^2} dN_s^{\mathrm{loc}} = -\int_0^t \left(\frac{C_s}{B_s}\right) d\left(\frac{B_s}{N_s}\right).$$

7.2 The Fubini condition

As usual, we must impose some L^1 conditions on the collateralized derivative $(C, \mathcal{M}, V)$. The first applies to the final settlement V,

$$\mathbb{E}_{\mathbb{N}}\left[\left|\frac{V}{N_T}\right|\right] < \infty. \tag{19}$$

This is well-known. Clearly, if it holds for $(N, \mathbb{N})$ then it holds for all equivalent numeraire pairs.

We also need to constrain the continuous collateral stream C. We insist on the following 'Fubini condition'. Define

$$F_t := \int_0^t \frac{C_s}{N_s}\left(\frac{dB_s}{B_s} - \frac{d\mathcal{M}_s}{\mathcal{M}_s}\right),$$

and let $V_t^{F:N}$ be the total variation of F_t. We require that

$$\mathbb{E}_{\mathbb{N}}\left[V_T^{F:N}\right] < \infty. \tag{20}$$

In the case when B is absolutely continuous with respect to Lebesgue measure,

$$B_t =: \exp\left(\int_0^t r_s ds\right),$$

this is equivalent to the requirement

$$\int_0^T \mathbb{E}_{\mathbb{N}}\left[\left|\frac{C_s}{N_s}(r_s - \mu_s)\right|\right] ds < \infty. \tag{21}$$

Written in this way it is clear that (21) is a natural analogue of (19).
It is not immediately clear that (20) is independent of the numeraire pair $(N, \mathbb{N})$ (and so is a restriction only on the derivative $(C, \mathcal{M}, V)$ and not on the numeraire pair $(N, \mathbb{N})$), but it is as the following lemma demonstrates.

Lemma 15. *Suppose the numeraire pairs* $(N, \mathbb{N})$ *and* $(\hat{N}, \hat{\mathbb{N}})$ *are equivalent and* (20) *holds. Then*

$$\mathbb{E}_{\hat{\mathbb{N}}}\left[V_T^{F:\hat{N}}\right] = \mathbb{E}_{\mathbb{N}}\left[V_T^{F:N}\right] < \infty.$$

Proof. This is an immediate consequence of Lemma A.1 in [8] (working in the measure $\mathbb{N}$ and taking, in their notation,

$$A_s := \int_0^s \left|\frac{C_u}{\hat{N}_u}\left(\frac{d\mathcal{M}_u}{\mathcal{M}_u} - \frac{dB_u}{B_u}\right)\right|, \qquad M_T := \frac{\hat{N}_T}{N_T},$$

for $0 \leq s \leq T$, and $\tau = T$). $\square$

Remark 16: When B is absolutely continuous with respect to Lebesgue measure this takes the form

$$\int_0^T \mathbb{E}_{\mathbb{N}}\left[\left|\frac{C_s}{N_s}(r_s - \mu_s)\right|\right] ds = \int_0^T \mathbb{E}_{\hat{\mathbb{N}}}\left[\left|\frac{C_s}{\hat{N}_s}(r_s - \mu_s)\right|\right] ds < \infty.$$

7.3 An explicit formula for V^C

We begin with the Doob-Meyer decomposition of the numeraire rebased value of V^C. We could have derived this decomposition earlier but it is more convenient and allows more insight to express it in terms of the process B.

Lemma 16. *With* B *defined as in Lemma 11 and* $R^{C:N}$ *defined as in equation* (18) *we can write the Doob-Meyer decomposition of* V^C/N *as*

$$\frac{V_t^C}{N_t} = \frac{C_0}{N_0} + \int_0^t \frac{C_s}{N_s}\left(\frac{d\mathcal{M}_s}{\mathcal{M}_s} - \frac{dB_s}{B_s}\right) + M_t^{C:N} - R_t^{C:N}. \tag{22}$$

Proof. Recall, from Section 5, that

$$M^{C:N} := \frac{V^C - C}{N} + \Phi^{C:N}$$

where

$$\Phi_t^{C:N} := \int_0^t \frac{\mathcal{M}_s}{N_s} \left(d\left(\frac{C_s}{\mathcal{M}_s}\right) - \frac{1}{N_s} d\left[\frac{C}{\mathcal{M}}, N\right]_s \right).$$

From Itô's formula (recall $\mathcal{M}$ and B are finite variation)

$$\begin{aligned}
\frac{\mathcal{M}_t}{N_t} d\left(\frac{C_t}{\mathcal{M}_t}\right)\left(1 - \frac{dN_t}{N_t}\right) &= \frac{C_t}{N_t}\left(\frac{dC_t}{C_t} - \frac{d\mathcal{M}_t}{\mathcal{M}_t}\right)\left(1 - \frac{dN_t}{N_t}\right) \\
&= \frac{C_t}{N_t}\left(\frac{dB_t}{B_t} - \frac{d\mathcal{M}_t}{\mathcal{M}_t}\right) + d\left(\frac{C_t}{N_t}\right) \\
&\qquad + \left(\frac{C_t}{N_t}\right)\left(\frac{dN_t}{N_t} - \frac{d[N]_t}{N_t^2} - \frac{dB_t}{B_t}\right).
\end{aligned}$$

Equation (12), the definition of B, now yields

$$\frac{\mathcal{M}_t}{N_t} d\left(\frac{C_t}{\mathcal{M}_t}\right)\left(1 - \frac{dN_t}{N_t}\right) = \frac{C_t}{N_t}\left(\frac{dB_t}{B_t} - \frac{d\mathcal{M}_t}{\mathcal{M}_t}\right) + d\left(\frac{C_t}{N_t}\right) + \left(\frac{C_t}{N_t^2}\right) dN_t^{\text{loc}},$$

and thus

$$M_t^{C:N} := \frac{V_t^C - C_t}{N_t} + \Phi_t^{C:N} = \frac{V_t^C}{N_t} - \frac{C_0}{N_0} + \int_0^t \frac{C_s}{N_s}\left(\frac{dB_s}{B_s} - \frac{d\mathcal{M}_s}{\mathcal{M}_s}\right) + R_t^{C:N},$$

from which the result follows. $\square$

This yields an explicit expression for V^C:

Corollary 17.

$$V_t^C := N_t \mathbb{E}_{\mathbb{N}}\left[\int_t^T \frac{C_s}{N_s}\left(\frac{d\mathcal{M}_s}{\mathcal{M}_s} - \frac{dB_s}{B_s}\right) \middle| \mathcal{F}_t\right] + N_t \mathbb{E}_{\mathbb{N}}\left[\frac{V}{N_T} \middle| \mathcal{F}_t\right]. \tag{23}$$

Proof. This follows immediately from equation (22) and the martingale property of $R^{C:N}$ and $M^{C:N}$. $\square$

An important consequence of (19) and (20) is that equation (23) holds when we change numeraire. The final term on the right-hand side of (23) is clearly (equivalent) numeraire-pair invariant. For the first term we need the following:

Lemma 18. *If* $(N, \mathbb{N})$ *and* $(\hat{N}, \hat{\mathbb{N}})$ *are equivalent numeraire pairs, and if* (20) *holds, then*

$$N_t \mathbb{E}_{\mathbb{N}}\left[\int_t^T \frac{C_s}{N_s}\left(\frac{d\mathcal{M}_s}{\mathcal{M}_s} - \frac{dB_s}{B_s}\right) \middle| \mathcal{F}_t\right] = \hat{N}_t \mathbb{E}_{\hat{\mathbb{N}}}\left[\int_t^T \frac{C_s}{\hat{N}_s}\left(\frac{d\mathcal{M}_s}{\mathcal{M}_s} - \frac{dB_s}{B_s}\right) \middle| \mathcal{F}_t\right].$$

As a result, equation (23) *holds for all equivalent numeraire pairs.*

Proof. This is an immediate consequence of Lemma A.1 in [8] (working in the measure $\mathbb{N}$ and taking, in their notation,

$$A_s := \int_0^s \frac{C_u}{\hat{N}_u}\left(\frac{d\mathcal{M}_u}{\mathcal{M}_u} - \frac{dB_u}{B_u}\right), \qquad M_T := \mathbf{1}_H \frac{\hat{N}_T}{N_T},$$

for $0 \le s \le T$, and $\tau = T, t$ for any arbitrary $H \in \mathcal{F}_t$). □

Remark 17: This shows that, if (23) holds for one numeraire pair, it holds for all equivalent numeraire pairs. As a consequence, it follows that $M^{C:N} - R^{C:N}$ is a martingale for all equivalent numeraire pairs, even if $M^{C:N}$ and $R^{C:N}$ are not individually martingales.

8 Pricing formulae for collateralized derivatives

8.1 Obtaining an expression for P

Lemma 19 below, which doesn't make any special assumptions on the economy, is the starting point for obtaining a general pricing formula for P, the price of a fully-collateralized derivative, when one exists.

Lemma 19. *With B defined as in Lemma 11 the process $PB/\mathcal{M}N$ is a local martingale with respect to* $(\{\mathcal{F}_t\}, \mathbb{N})$.

Proof. By Itô's formula

$$d\left(\frac{P_tB_t}{\mathcal{M}_tN_t}\right) = \frac{B_t}{N_t}d\left(\frac{P_t}{\mathcal{M}_t}\right) + \frac{P_t}{\mathcal{M}_t}d\left(\frac{B_t}{N_t}\right) - d\left(\frac{P_t}{\mathcal{M}_t}\right)\frac{B_tdN_t}{N_t^2}. \tag{24}$$

If we now recall that the martingale $M^{P:N}$ introduced in Section 5 satisfies (equation (8)),

$$dM_t^{P:N} = d\Phi_t^{P:N} = \frac{\mathcal{M}_t}{N_t}d\left(\frac{P_t}{\mathcal{M}_t}\right) - d\left(\frac{P_t}{\mathcal{M}_t}\right)\frac{\mathcal{M}_tdN_t}{N_t^2},$$

we can write (24) as

$$d\left(\frac{P_tB_t}{\mathcal{M}_tN_t}\right) = \frac{B_t}{\mathcal{M}_t}d\Phi_t^{P:N} + \frac{P_t}{\mathcal{M}_t}d\left(\frac{B_t}{N_t}\right) \tag{25}$$

and the result follows as we have expressed our process as a sum of integrals against local martingales. □

Although the local martingale property holds generally for the process $PB/\mathcal{M}N$, given any particular economy, payoff V and collateral rate μ, it may be that it is not a martingale. This is the case even if the economy is complete and we know that we can price the collateralized derivative. If we assume for the economy that the random variable $VB_T/\mathcal{M}_TN_T$ is integrable and so, via conditioning on the filtration $\{\mathcal{F}_t\}$, it can be used to form a martingale which agrees with the local martingale $PB/\mathcal{M}N$ at time T, it cannot be assumed that the two processes are equal—it maybe the case that $PB/\mathcal{M}N$ is a local martingale but not a martingale. We give an example of this in Example 2 below. The corollary below indicates in the complete case what needs to be checked before the simple pricing formula suggested by the lemma above can be applied. In an incomplete economy one would need to check that an admissible trading strategy existed before applying any pricing formula.

The following corollary now follows immediately from the discussion in Section 5.

Corollary 20. *Suppose that the process B defined in Lemma 11 with repect to the numeraire pair $(N, \mathbb{N})$ is a generalized price process and that $VB_T/\mathcal{M}_T N_T$ is in $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{N})$. Define the process P via*

$$P_t := \frac{\mathcal{M}_t N_t}{B_t} \mathbb{E}_{\mathbb{N}} \left[\frac{VB_T}{\mathcal{M}_T N_T} \middle| \mathcal{F}_t \right]. \tag{26}$$

If

$$M_t^{P:N} := \Phi_t^{P:N} := \int_0^t \frac{\mathcal{M}_s}{N_s} \left(d\left(\frac{P_s}{\mathcal{M}_s}\right) - \frac{1}{N_s} d\left[\frac{P}{\mathcal{M}}, N\right]_s \right)$$

is a martingale with respect to $(\{\mathcal{F}_t\}, \mathbb{N})$ which can be replicated by an admissible strategy that invests income received from the collateral in the numeraire N and further

$$R_t^{P:N} = \int_0^t \frac{P_s}{N_s^2} dN_s^{\mathrm{loc}}.$$

is a martingale with respect to $(\{\mathcal{F}_t\}, \mathbb{N})$, then the process P can be interpreted as the collateral process of the collateralized trade which pays V at time T, and P_t is its corresponding price at time t.

In practice there will always be a full-collateral process P for which $PB/\mathcal{M}N$ will be a true martingale as the following result shows.

Corollary 21. *Suppose that $\mathcal{M}_t/B_t$, $t \in [0, T]$, is of bounded variation, almost surely, and that B/N is a martingale with respect to $(\{\mathcal{F}_t\}, \mathbb{N})$ and the economy is complete. Then P defined via equation* (26) *can be interpreted as the collateral process for the fully-collateralized trade which pays V at time T, and here $PB/\mathcal{M}N$ is a martingale.*

Proof. Define P via equation (26), meaning $PB/\mathcal{M}N$ is a martingale under $\mathbb{N}$.

As B/N is a martingale we can use it as the Radon-Nikodym derivative to move to a measure $\mathbb{B}$ in which $P/\mathcal{M}$ is a martingale. We can apply Corollary 20 using the numeraire pair $(B, \mathbb{B})$. Here $R^{P:B} = 0$ and so is trivially a martingale. Using B as numeraire, (25) becomes

$$d\left(\frac{P_t}{\mathcal{M}_t}\right) = \frac{B_t}{\mathcal{M}_t} d\Phi_t^{P:B}.$$

By definition (equation (26)) $PB/\mathcal{M}N$ is a martingale under $\mathbb{N}$, so $P/\mathcal{M}$ is a martingale under $\mathbb{B}$. By assumption, $\mathcal{M}/B$ is of bounded variation, so it follows from Lemma A.1 in [8] that $\Phi^{P:B}$ is also a martingale under $\mathbb{B}$. As the economy is complete the (gain) process martingale $\Phi^{P:B}$ can be replicated by an admissible trading strategy, meaning that our choice P is indeed a full-collateral process for our collateralized derivative. □

As noted earlier in our analysis we have not had to assume the existence of a risk-free rate. Further we cannot always move to an equivalent martingale measure corresponding to B as numeraire unless B is a price process. Finally, note that even if the claim V paid at T can be valued in the original economy, unless the agreed collateral rate μ is chosen appropriately it may be the case that we cannot price the corresponding collateralized derivative.

We conclude this section with an example which shows that the process P is not necessarily unique and, in particular, that a full-collateral process can exist for which the process $PB/\mathcal{M}N$ is only a local martingale. This is in contrast to the discrete setup considered in Section 4 (there are no discrete-time local martingales).

Example 2: Consider the following simple economy, defined for $0 \le t \le 1$. There are two assets. The first, B, is a finite variation bond. This is, in fact, deterministic and constant, $B_t \equiv 1$ for all t.
The second asset is a lognormal martingale, with unit volatility:

$$dS_t := S_t dW_t, \qquad S_t = \exp\left(W_t - \tfrac{1}{2}t\right).$$

This economy is one of the simplest and nicest possible; there is a finite variation asset, the economy is complete and it is arbitrage-free.
Now let $\sigma : [0, 1) \to \mathbb{R}$ be some positive function such that $\Sigma_t < \infty$ for $t < 1$ and $\Sigma_1 = \infty$ where

$$\Sigma_t^2 := \int_0^t \sigma_s^2 ds.$$

Now define the process X for $t < 1$ via

$$dX_t = \sigma_t dW_t, \qquad X_0 = 0,$$

and let τ be the stopping time $\tau := \inf\{t > 0 : X_t = 1\}$ .Note that $\tau < 1$ almost surely, and define Y by

$$\begin{aligned} \hat{\sigma}_t &:= \sigma_t \mathbf{1}\{t < \tau\} + \sigma_\tau \mathbf{1}\{t \ge \tau\} \\ dY_t &:= \hat{\sigma}_t dW_t, \qquad Y_0 = 0. \end{aligned}$$

Now note that $\mathbb{E}[Y_\tau] = 1 \neq 0 = Y_0$,thus Y is a strict local martingale.
Now we can set up our collateralized derivative. The derivative finally settles at time one. The collateral rate is defined via $\mathcal{M}_t := \hat{\sigma}_t^{-1}$. We define V, the derivative's final payoff, as $V := Y_1/\hat{\sigma}_1$. and claim that $M_t^{P:B} (= \Phi_t^{P:B}) = W_t$ and $P_t = Y_t/\hat{\sigma}_t$ is a solution. To see this, note that with these definitions $P_1 = Y_1/\hat{\sigma}_1 = V$, as required, and

$$d\left(\frac{P_t}{\mathcal{M}_t}\right) = d\left(\frac{Y_t/\hat{\sigma}_t}{\mathcal{M}_t}\right) = dY_t = \hat{\sigma}_t dW_t = \frac{1}{\mathcal{M}_t} dM_t^{P:B},$$

which is equation (8), as required.
Note that $M^{P:B}$ is a true martingale, as required, but $P/\mathcal{M}$ is only a local martingale but not a true martingale. Furthermore, the martingale condition for $R^{P:B}$ defined in equation (18) follows trivially since $N = B$ is finite variation, and one can check that the Fubini condition (21) holds for a suitable choice of Σ^2, such as

$$\Sigma_t^2 := \begin{cases} t, & t \le \pi/8, \\ \frac{\pi}{8} \exp\left(\frac{(1-t)^{-\beta} - (1-\pi/8)^{-\beta}}{\beta}\right), & t > \pi/8, \end{cases}$$

for $0 < \beta < 1$.
This is not the only solution. For example, there is another solution $\hat{P}$ in which $\hat{P}/\mathcal{M}$ is a martingale. Define $\hat{P}_1 := Y_1/\hat{\sigma}_1$, and

$$\begin{aligned} \frac{\hat{P}_t}{\mathcal{M}_t} &:= \mathbb{E}_{\mathbb{B}}\left[\left.\frac{\hat{P}_1}{\mathcal{M}_1}\right| \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{B}}\left[Y_1 | \mathcal{F}_t\right] = 1 + \sigma_\tau \left(W_t - W_\tau\right) \mathbf{1}\{t \ge \tau\}, \\ \hat{P}_t &= \mathcal{M}_t + \left(W_t - W_\tau\right) \mathbf{1}\{t \ge \tau\}, \\ \hat{M}_t^{\hat{P}:B} &:= \int_0^t \mathcal{M}_s d\left(\frac{\hat{P}_s}{\mathcal{M}_s}\right) = \mathbf{1}\{t \ge \tau\} \int_\tau^t dW_s = \left(W_t - W_\tau\right) \mathbf{1}\{t \ge \tau\} = W_t - W_{t \wedge \tau}. \end{aligned}$$

$\hat{M}^{\hat{P}:B}$ is a martingale. As in the strict local martingale example above, (18) defines a martingale since $N = B$ is finite variation, and the Fubini condition (21) holds.

Remark 18: In this example we have presented two distinct solutions. But any convex combination of these two solutions is also a solution, hence there are infinitely many solutions.

Remark 19: Although, in the continuous-cashflow setting, there are situations such as Example 2, in which $PB/\mathcal{M}N$ is a strict local martingale, the convergence theorem in Section 9, Theorem 24, shows that in practice we are only interested in cases for which $PB/\mathcal{M}N$ is a true martingale.

8.2 Partial collateralization

As promised in Section 5, now that we have introduced the process B we return to consider the partially-collateralized case introduced there. The basic idea is to turn a partially-collateralized derivative into a fully-collateralized one and then use the results already derived for fully-collateralized derivatives. We begin by observing that collateral posted at the 'rate' of the 'risk-free account' B has no effect on the value of a collateralized trade. This result is summarized in the following Lemma.

Lemma 22. *Consider a collateralized derivative with collateral stream* C*, final redemption* $V = 0$ *and in which all the collateral is invested at the rate of the 'risk-free account'* B *(i.e.,* $\mathcal{M} = B$*). Assuming the resulting derivative can be replicated by some admissible trading strategy, that invests income received from the collateral in the numeraire* N *and* $R^{C:N}$ *is a martingale under* $\mathbb{N}$ *then the process* V^C/N *is a martingale under* $\mathbb{N}$ *and we have* $V_t^C = 0$.

Proof. This follows immediately from (23). □

Remark 20: In general it may not be possible to invest income from the collateral stream in B as it may not be a generalized price process.

Armed with Lemma 22 we can now address the question of a partially-collateralized derivative. Recall that α is some $\{\mathcal{F}_t\}$-adapted process and the amount of collateral posted at time t is given by $C_t := \alpha_t V_t^C$. In general α may depend on the process V^C. Making the obvious generalization of notation from the fully-collateralized case, we define P_t^α to be the time-t value of a partially-collateralized derivative entered at time t. In this case, equation (8), becomes

$$M_t^{\alpha P^\alpha:N} = \frac{V_t^{\alpha P^\alpha} - \alpha_t P_t^\alpha}{N_t} + \Phi_t^{\alpha P^\alpha:N} = \frac{(1-\alpha_t)\, P_t^\alpha}{N_t} + \Phi_t^{\alpha P^\alpha:N}. \tag{27}$$

If we can find a process P^α such that $M^{\alpha P^\alpha:N}$ is a martingale then this is the price of the partially-collateralized derivative (note, that as in the fully-collateralized case, this may not be unique). The way we do this is by identifying a fully-collateralized derivative with price process P^α. This derivative is constructed by starting with the partially-collateralized derivative and 'investing the rest of the collateral in B', which does not affect the valuation, as Lemma 22 demonstrated.

First we extend our earlier notation. Recall that V^C denotes the value of a collateralized derivative with collateral stream C, collateral interest rate μ and final settlement V (the μ and V are suppressed in the notation). Paying the interest-rate μ corresponds to investing the collateral in a bank account

$$\mathcal{M}_t := \exp\left(\int_0^t \mu_s ds\right).$$

Now, for any $\{\mathcal{F}_t\}$-adapted process α, we define $V^{\alpha:C}$ to be the value of a collateralized derivative with collateral stream C, collateral interest corresponding to investing the collateral in the bank account $\mathcal{M}^\alpha$ and final settlement V where

$$\mathcal{M}_t^\alpha := \exp\left(\int_0^t \alpha_s \frac{d\mathcal{M}_s}{\mathcal{M}_s} + (1-\alpha_s)\frac{dB_s}{B_s}\right). \tag{28}$$

We also extend our definition of $\Phi^{C:N}$ similarly:

$$\Phi_t^{\alpha:C:N} := \int_0^t \frac{\mathcal{M}_s^\alpha}{N_s} d\left(\frac{C_s}{\mathcal{M}_s^\alpha}\right)\left(1 - \frac{dN_s}{N_s}\right).$$

Note that $V^{1:C} = V^C$, $\mathcal{M}^0 = B$, $\mathcal{M}^1 = \mathcal{M}$ and $\Phi^{1:C:N} = \Phi^{C:N}$.

Remark 21: We have needed to be a little clumsy here in the way we have described the interest paid on the collateral. This is because B may not, in general, be absolutely continuous with respect to Lebesgue measure. When it is we can write

$$B_t =: \exp\left(\int_0^t r_s\, ds\right), \qquad \mu_t^\alpha := \alpha_t \mu_t + (1-\alpha_t) r_t, \qquad \mathcal{M}_t^\alpha := \exp\left(\int_0^t \mu_s^\alpha\, ds\right),$$

and so interest is paid on the collateral at the rate μ^α. Let $\hat{P}^\alpha$ be the price of the fully-collateralized derivative with $\mathcal{M}^\alpha$. We show that $P^\alpha := \hat{P}^\alpha$ solves (27), so is the value of the partially-collateralized derivative.

Theorem 23. *The price process $\hat{P}^\alpha$ of the fully-collateralized derivative with account $\mathcal{M}^\alpha$ makes*

$$\hat{M}^{\alpha \hat{P}^\alpha:N} := \frac{(1-\alpha)\,\hat{P}^\alpha}{N} + \Phi^{\alpha \hat{P}^\alpha:N} \tag{29}$$

a martingale. Thus $P^\alpha := \hat{P}^\alpha$ is also the price of the partially-collateralized derivative in which the collateral posted is αP^α and interest is paid on collateral at rate μ.

Proof. Consider the collateralized derivative in which $C_t := (1-\alpha_t)\hat{P}_t^\alpha$ and interest is paid according to B. By Lemma 22

$$\Phi^{0:(1-\alpha)\hat{P}^\alpha:N} - \frac{(1-\alpha)\,\hat{P}^\alpha}{N} \tag{30}$$

is a martingale. Furthermore, since $\hat{P}^\alpha$ is the price of a fully-collateralized derivative with collateral account $\mathcal{M}^\alpha$, $\Phi^{\alpha:\hat{P}^\alpha:N}$ is also a martingale. Subtracting (30) from $\Phi^{\alpha:\hat{P}^\alpha:N}$ now shows that

$$\frac{(1-\alpha)\,\hat{P}^\alpha}{N} + \left(\Phi^{\alpha:\hat{P}^\alpha:N} - \Phi^{0:(1-\alpha)\hat{P}^\alpha:N}\right)$$

is a martingale, so to complete the proof that (29) is a martingale, it suffices to prove that

$$\Phi^{\alpha:\hat{P}^\alpha:N} = \Phi^{1:\alpha\hat{P}^\alpha:N} + \Phi^{0:(1-\alpha)\hat{P}^\alpha:N}. \tag{31}$$

What this means, in words, is that the following two strategies lead to exactly the same holding. Under the first, a collateralized derivative is entered with collateral $\hat{P}$ and interest rate $\mathcal{M}^\alpha$. As cashflows are received they are invested in the numeraire N. Under the second strategy, two collateralized derivatives are entered. The first has collateral $\alpha\hat{P}$ and interest rate $\mathcal{M}$; the second has collateral $(1-\alpha)\hat{P}$ and interest rate B. Again all cashflows are invested in the numeraire N.

Examining the definitions of $\Phi_t^{\alpha:\hat{P}^\alpha:N}$, $\Phi_t^{1:\alpha\hat{P}^\alpha:N}$ and $\Phi_t^{0:(1-\alpha)\hat{P}^\alpha:N}$, to prove (31), it suffices to show that

$$\mathcal{M}_t^\alpha d\left(\frac{\hat{P}_t^\alpha}{\mathcal{M}_t^\alpha}\right) = \mathcal{M}_t d\left(\frac{\alpha_t\hat{P}_t^\alpha}{\mathcal{M}_t}\right) + B_t d\left(\frac{(1-\alpha_t)\hat{P}_t^\alpha}{B_t}\right). \tag{32}$$

But it follows from (28), the definition of $\mathcal{M}^\alpha$, that

$$\frac{d\mathcal{M}_t^a}{\mathcal{M}_t^a} = \alpha_t\frac{d\mathcal{M}_t}{\mathcal{M}_t} + (1-\alpha_t)\frac{dB_t}{B_t},$$

and the result now follows by applying Itô's formula to (32). ☐

9 Convergence

In common with other authors we have primarily analysed the continuous-cashflow model of Section 5 rather than the discrete model of Section 4, as it is both easier to analyse and provides more insight. But this is only worthwhile if the continuous model is, in fact, a good approximation to the discrete model it is intended to mirror.

We saw in Section 4 that in the discrete-cashflow case that the price of a fully-collateralized derivative is completely characterised by the distribution of X_T^n and the fact that X^n is a martingale, where

$$X_t^n := \frac{P_t^n}{N_t}\frac{B_t^n}{\mathcal{M}_t^n}.$$

By contrast, we saw in Section 5 that, in the continuous-cashflow case, the corresponding process

$$X_t := \frac{P_t}{N_t}\frac{B_t^n}{\mathcal{M}_t},$$

is only a local martingale, not necessarily a true martingale. As we saw in Section 8, this does not uniquely characterise P.

Below we provide a convergence result that addresses both of these issues. It shows that the continuous-cashflow model is a good approximation to the discrete-cashflow model and, importantly, that we should in practice consider only the solution P for which X is a true martingale and not any strict local martingale solutions.

Theorem 24. *Define*

$$X_t^n := \frac{P_t^n}{N_t}\frac{B_t^n}{\mathcal{M}_t^n}$$

and suppose that, as $n\to\infty$, $X_T^n \to X_T$ in $\mathcal{L}^p$ for some random variable X_T, some $p\geq 1$. Then, for any $t\in[0,T]$, $X_t^n\to X_t$ in $\mathcal{L}^p$ as $n\to\infty$, where

$$X_t := \mathbb{E}_{\mathbb{N}}\left[X_T|\,\mathcal{F}_t\right].$$

Furthermore, if $p > 1$, then

$$\sup_{t\in[0,T]} |X^n_t - X_t| \to 0,$$

in $\mathcal{L}^p$, as $n \to \infty$.

Proof. These results are immediate consequences of the martingale properties of X^n and X. The first result is a standard application of the conditional version of Jensen's equality.The second result follows from Doob's $\mathcal{L}^p$ inequality (see, for example, Theorem 3.25 in [9]) applied to the martingale $X^n - X$. □

Theorem 24 is a strong convergence result for the processes X^n. Exactly what this means for the convergence of the processes P^n depends on exactly how the processes $B^n/\mathcal{M}^n$ converge. Here is one result:

Corollary 25. *Fix $t \in [0, T]$, and recall that $P^n_T = V$. Suppose that, as $n \to \infty$,*

$$\frac{P^n_T}{N_T}\frac{B^n_T}{\mathcal{M}^n_T} \to \frac{V}{N_T}\frac{B_T}{\mathcal{M}_T} \qquad \textit{in } \mathcal{L}^1,$$

that

$$\frac{B^n_t}{\mathcal{M}^n_t} \to \frac{B_t}{\mathcal{M}_t} \qquad \textit{in probability,}$$

and define

$$P_t := \frac{\mathcal{M}_t N_t}{B_t}\mathbb{E}_{\mathbb{N}}\left[\frac{V}{N_T}\frac{B_T}{\mathcal{M}_T}\middle|\mathcal{F}_t\right].$$

Then $P^n_t \to P_t$ in probability as $n \to \infty$.

Proof. Note that

$$P^n_t = \left(\frac{\mathcal{M}^n_t N_t}{B^n_t}\right) X^n_t.$$

Further $\mathcal{M}^n_t N_t/B^n_t \to \mathcal{M}_t N_t/B_t$ in probability since $B^n_t/\mathcal{M}^n_t \to B_t/\mathcal{M}_t$ in probability. Also $X^n_t \to X_t$ in probability since (Theorem 24) $X^n_t \to X_t$ in $\mathcal{L}^1$. This is enough to prove our result. □

10 A direct replication of P

A number of authors have approached the problem in the continuous-cashflow setting of finding a formula for the price, P, of a fully-collateralized derivative by starting their analysis from a replication argument. Replication is the usual starting point for the PDE approach to pricing. Piterbarg [11] was one of the first to explore this route for partially collateralized derivatives in the Black-Scholes framework. Brigo *et al.* [2] clarified the self-financing condition for the replicating portfolio used in [11] and Han *et al.* [7] (amongst others) generalized the approach from the Black-Scholes case. The replicating portfolio considered in these papers takes the collateral account as one of the 'underlying assets' in the economy. This approach is a break from the classical theory—the process P, rebased by the numeraire, is not a martingale under the equivalent martingale measure—and it means that the close link between an admissible trading strategy and the martingale approach to pricing won't hold (without modification).

It is interesting to revisit the replication approach and see how it fits in with the analysis in this paper when we keep within the classical theory of what constitutes a self-financing strategy. Unusually in the application here, for some cases at least, a judicous choice of replicating strategy when combined with the earlier martingale result gives a direct route to the pricing formula Corollary 20.

We will consider the situation in which B is a generalized price process and the collateralized derivative can be replicated. Consider the following simple trading strategy:

- Start with some initial wealth V_0;
- At any time t, hold an amount $B_t/\mathcal{M}_t$ of the collateralized trade;
- Invest all the proceeds, so far, of this strategy plus the initial wealth V_0 in B.

It is clear that this is a self-financing strategy.

It turns out that V_t, the value of our portfolio at time t is precisely $P_tB_t/\mathcal{M}_t$. To see this, note first that for the strategy (α_t, β_t) in which, at time t, we hold α_t units of the collateralized trade and β_t units of the cash account, the gain, G_t, is given by

$$G_t = \int_0^t \alpha_u \mathcal{M}_u d\left(\frac{P_u}{\mathcal{M}_u}\right) + \int_0^t \beta_u dB_u.$$

Recall that in the fully-collateralized case the value of future cashflows, $V_t^P - P_t$, is zero hence there is no contribution in G_t from the value of the holding of α_t units of the fully-collateralized derivative, and all our wealth is invested in B. It follows that $\beta_t = V_t/B_t$, and in our case we have chosen $\alpha_t = B_t/\mathcal{M}_t$.

As our initial portfolio has value V_0, the portfolio's value at t, V_t is given by, $V_t = G_t + V_0$. Putting this together yields

$$\begin{aligned} V_t &= V_0 + \int_0^t \alpha_u \mathcal{M}_u d\left(\frac{P_u}{\mathcal{M}_u}\right) + \int_0^t \beta_u dB_u \\ &= V_0 + \int_0^t B_u d\left(\frac{P_u}{\mathcal{M}_u}\right) + \int_0^t \frac{V_u}{B_u} dB_u, \qquad (33) \\ dV_t &= B_t d\left(\frac{P_t}{\mathcal{M}_t}\right) + V_t \frac{dB_t}{B_t}. \qquad (34) \end{aligned}$$

Now observe, from Itô's formula, that

$$d\left(\frac{P_t}{\mathcal{M}_t} B_t\right) = B_t d\left(\frac{P_t}{\mathcal{M}_t}\right) + \left(\frac{P_t}{\mathcal{M}_t} B_t\right) \frac{dB_t}{B_t}. \qquad (35)$$

If we define $\Delta_t := V_t - P_tB_t/\mathcal{M}_t$, then subtracting (35) from (34) yields

$$d\Delta_t = \Delta_t \frac{dB_t}{B_t} = \Delta_t d\log(B_t).$$

Thus Δ is the Doléans exponential of $\log(B)$, and

$$\Delta_t = \Delta_0 \frac{B_t}{B_0}. \qquad (36)$$

Finally, choosing $V_0 = P_0$ means $\Delta_0 = 0$ and thus (36) implies that, for all $t \geq 0$,

$$\Delta_t = 0, \qquad V_t = \frac{P_t}{\mathcal{M}_t} B_t,$$

and (33) becomes

$$\frac{P_t B_t}{\mathcal{M}_t} = P_0 + \int_0^t B_u d\left(\frac{P_u}{\mathcal{M}_u}\right) + \int_0^t \frac{P_t}{\mathcal{M}_t} dB_u = P_0 + \int_0^t \frac{B_u}{\mathcal{M}_u} dI_u^P + \int_0^t \frac{P_t}{\mathcal{M}_t} dB_u,$$

where

$$I_t^P := \int_0^t \mathcal{M}_u d\left(\frac{P_u}{\mathcal{M}_u}\right).$$

If we now rebase the above equation in terms of the numeraire N we obtain (applying Itô's formula)

$$\left(\frac{P_t B_t}{\mathcal{M}_t N_t}\right) = \frac{P_0}{N_0} + \int_0^t \frac{B_u}{\mathcal{M}_u} d\Phi_u^{P:N} + \int_0^t \frac{P_u}{\mathcal{M}_u} d\left(\frac{B_u}{N_u}\right).$$

which is just equation (25), the last relation in the proof of Lemma 19. If the rebased gain process on the right hand side is a martingale for the particular economy under consideration then from this equation we can see that, provided $\Phi^{P:N}$ is a martingale under $\mathbb{N}$ and B is a generalized price process, there will exist an admissible trading strategy in terms of the underlying assets A for replicating $PB/\mathcal{M}$ and the pricing formula will apply. For a price process P for which $PB/\mathcal{M}N$ is only a local martingale we cannot find an admissible replicating strategy.

Remark 22: If $(\hat{N}, \hat{\mathbb{N}})$ is a different numeraire pair for the economy the assumption that $\Phi^{P:N}$ is a martingale under $\mathbb{N}$ does not mean $\Phi^{P:\hat{N}}$ must be a martingale under $\hat{\mathbb{N}}$. However under the assumptions here the pricing formula using the alternative numeraire pair will still hold. In order for the collateralized trade to be replicable we must have some numeraire pair $(N, \mathbb{N})$ for which $\Phi^{P:N}$ is a martingale. We will elaborate on this point in [10].

11 Cooking with collateral

The viewpoint we have taken in this paper so far is that we start with an economy with underlying assets A and a numeraire pair. We then specified a collateral rate μ and considered the problem of pricing a collateralized derivative associated with a trade having a payoff V at some future time. In this situation, in order to be able to price any given collateralized trade we must be able to replicate the collateral cashflow stream using the underlying assets A in our economy.

The title of this section, *Cooking with Collateral*, is taken from a paper by Piterbarg [12] in which an alternative viewpoint is taken: a collection of collateralized derivatives is taken as the primitive for the economy and the dynamics of these underlying assets are given under the real world measure $\mathbb{P}$. No other information on the economy is specified. Indeed the purpose of Piterbarg's article is to develop a model for an economy without introducing a risk-free rate and with all assets traded on a collateralized basis. In this section we explore this viewpoint in the light of our previous analysis and review what can and cannot be known about the economy based on the information given. We will show that Piterbarg's 'economy' is more standard than it might at first appear—it just hasn't been fully specified.

It is not possible to have an economy containing only fully-collateralized derivatives. To enter one of these derivatives there is no net cashflow, $V_t^P - P_t = 0$, and the value of all future contractual cashflows is always zero. So there must be at least one other asset in the economy, one that has a non-zero value. Once we have identified this asset and explicitly specified its dynamics we can formalize what is meant by an arbitrage for this economy, and this leads naturally to the necessary condition for no arbitrage that Piterbarg proposes. Further we can then use this 'valuable' asset and the collateralized derivatives to create a strictly positive finite variation generalized price process, B. It then follows that the measure $\mathbb{B}$ identified by Piterbarg is such that $(B, \mathbb{B})$ is a numeraire pair for the economy.

We conclude with a result that summarizes conditions for this measure to be able to be interpreted as an equivalent martingale measure for the economy. This involves a modification of the classical definition.

In what follows we will focus our discussion on Piterbarg's first example where he considers just two collateralized assets. This example is sufficient to demonstrate key points and the more general case is a straightforward extension. In this example the collateralized assets are both fully-collateralised and interest is paid on both at the same rate μ, which is expressly not assumed to be deterministic, and the dynamics of the price processes of the collateral streams P^1 and P^2 associated with these assets are driven by a single Brownian motion and given by processes under the real world measure $\mathbb{P}$ as follows:

$$dP_t^i = \eta_t^i P_t^i dt + \sigma_t^i P_t^i dW_t, \; i = 1, 2. \tag{37}$$

Note that (37) is written in 'lognormal form', but Piterbarg does not consider only the case when η^i and σ^i are deterministic, so the dynamics of the P^i can be much more general than lognormal.

11.1 The collateralized assets with prices (P^1, P^2) do not form a fully-defined economy

We begin with a key observation. As noted above, though the collateralized trades can be taken as underlying assets they cannot make up the entire economy. There must be at least one other asset in the economy which has value and where cashflows from the collateralized trades can be invested. Either this asset must be positive always, or there must be more than one valuable asset in the economy such that a numeraire (positive generalized price process) can be constructed. This positivity property holds as otherwise there could be times when one had nowhere to invest income from the collateralized trades. We will denote this (numeraire) asset by N.

Of course, Piterbarg did not specify the dynamics of N, and they could be quite general. We will suppose that N satisfies:

$$dN_t = dF_t + \hat{\sigma}_t dW_t,$$

for some finite variation process F and some predictable process $\hat{\sigma}$. Note that we have chosen to use the same Brownian driver W for N as Piterbarg has for his P^1 and P^2. We could have chosen a different driver, introducing a second Brownian motion, but this would extend the economy in a way Piterbarg had not intended.

Note the meaning we attach to the phrase 'not fully specified'. Obviously, even for a standard economy, it is not possible or desirable to formulate a model including all assets that exist. But a model does need to be one that could, theoretically, exist. So, for example, if we were pricing an interest-rate derivative we would choose not to model equities explicitly in our economy, but an economy comprising only interest-rate products could exist theoretically. So, in this sense, our interest-rate model is fully specified. By contrast, an economy comprising only collateralized derivatives cannot exist—there must be at least one more asset. We need to include it explicitly so we can, for example, check there is no arbitrage.

11.2 Conditions for no arbitrage

Piterbarg begins by considering a trading strategy which goes long a notional of $\sigma_t^2 P_t^2$ in the first collateralized asset and short a notional of $\sigma_t^1 P_t^1$ in the second collateralized asset. By considering the gain process from this strategy in the two collateralized assets, he identifies a finite variation process which he claims must be zero, otherwise the economy will admit arbitrage. The claim that this finite variation process must be zero is correct because of the following lemma.

Lemma 26. *Let $\mathcal{E}$ be an economy comprising the collateralized trades with prices (P^1, P^2) and (at least one) other asset(s), admits a numeraire pair $(\hat{N}, \hat{\mathbb{N}})$ (a condition slightly stronger than arbitrage-free [3]), and consider the trading strategy φ in which, at time t, we hold:*

- $\varphi_t^1 := \sigma_t^2 P_t^2$ *units of the collateralized asset with price* P^1,
- $\varphi_t^2 := -\sigma_t^1 P_t^1$ *units of the collateralized asset with price*P^2, *and*
- $\varphi_t^{\hat{N}}$ *units of* $\hat{N}$, *where* $\varphi_0^{\hat{N}} = 0$.

Then for this strategy to satisfy the self-financing condition we must have $\varphi_t^{\hat{N}} = 0$, *all t.*

Proof. The holding of the numeraire at any time t comes from the income arising from the collateralized assets, thus

$$
\begin{aligned}
\varphi_t^{\hat{N}} &= \int_0^t \varphi_u^1 d\Phi_u^{P^1:\hat{N}} + \int_0^t \varphi_u^2 d\Phi_u^{P^2:\hat{N}} = \int_0^t \sigma_u^2 P_u^2 d\Phi_u^{P^1:\hat{N}} - \int_0^t \sigma_u^1 P_u^1 d\Phi_u^{P^2:\hat{N}} \\
&= \int_0^t \left(\sigma_u^2 P_u^2 d\left(\frac{P_u^1}{\mathcal{M}_u}\right) - \sigma_u^1 P_u^1 d\left(\frac{P_u^2}{\mathcal{M}_u}\right)\right) \frac{\mathcal{M}_u}{\hat{N}_u}\left(1 - \frac{d\hat{N}_u}{\hat{N}_u}\right).
\end{aligned}
$$

Substituting in from equation (37) we see that $\varphi^{\hat{N}}$ is a finite variation process.
But, as the collateralized assets contribute zero value, the value of our portfolio at any time t, the gain from the trading strategy, is just $\varphi_t^{\hat{N}} \hat{N}_t$. This, when divided by the numeraire $\hat{N}$ will, therefore, be a local martingale under the measure $\hat{\mathbb{N}}$. So $\varphi^{\hat{N}}$ is a local martingale under $\hat{N}$, and we already know it is of finite variation.
But a continuous finite variation local martingale is constant hence, for all t, $\varphi_t^{\hat{N}} = \varphi_0^{\hat{N}} = 0$ as required. ☐

Corollary 27. *A necessary condition for the economy $\mathcal{E}$, containing both the collateralized assets with prices P^1 and P^2, to be arbitrage-free is that*

$$\sigma_t^2(\eta_t^1 - \mu_t) = \sigma_t^1(\eta_t^2 - \mu_t),$$

for all t.

Proof. This follows directly from the requirement that the numeraire holding $\varphi^{\hat{N}}$ must be zero,

$$\int_0^t \sigma_u^2 P_u^2 d\Phi_u^{P^1:\hat{N}} - \int_0^t \sigma_u^1 P_u^1 d\Phi_u^{P^2:\hat{N}} = 0. \tag{38}$$

☐

Remark 23: Note that the condition (38) does not depend on the numeraire $\hat{N}$.

11.3 The cash account does exist

Piterbarg has avoided explicitly introducing the cash account and the risk-free rate. But it is in his economy, as we now demonstrate.

Theorem 28. *The economy $\mathcal{E}$ contains a finite variation (generalized) price process (the cash account), namely*

$$B_t \,=\, \exp\left(\int_0^t \frac{dF_s}{N_s} - \frac{\hat{\sigma}_s}{N_s} J_s ds\right),$$

where

$$J_t \,:=\, \frac{\eta_t^1 - \mu_t}{\sigma_t^1}.$$

Proof. Consider a trading strategy in which we hold φ^1 of the first collateralized derivative (which has price P^1), zero of the second (which has price P^2), and $\hat{\varphi}$ of the numeraire N, and define $\varphi := (\varphi^1, 0, \hat{\varphi})$. The gain process corresponding to this strategy is

$$G_t \,=\, G_0 + \int_0^t \varphi_s^1 \mathcal{M}_s d\left(\frac{P_s^1}{\mathcal{M}_s}\right) + \int_0^t \hat{\varphi}_s dN_s,$$

whence

$$\begin{aligned} dG_t \,&=\, \varphi_t^1 \mathcal{M}_t d\left(\frac{P_t^1}{\mathcal{M}_t}\right) + \hat{\varphi}_t dN_t, = \varphi_t^1 (dP_t^1 - \mu_t P_t^1 dt) + \hat{\varphi}_t dN_t \\ &=\, \left\{\varphi_t^1 (\eta_t^1 - \mu_t) P_t^1 dt + \hat{\varphi}_t dF_t\right\} + \left(\varphi_t^1 \sigma_t^1 P_t^1 + \hat{\varphi}_t \hat{\sigma}_t\right) dW_t. \end{aligned} \tag{39}$$

If we now insist that

$$\varphi_t^1 \,=\, -\frac{\hat{\sigma}_t}{\sigma_t^1 P_t^1} \hat{\varphi}_t,$$

then (39) becomes

$$dG_t \,=\, \varphi_t^1 (\eta_t^1 - \mu_t) P_t^1 dt + \hat{\varphi}_t dF_t,$$

which is of finite variation.
For the strategy $\varphi = (\varphi^1, 0, \hat{\varphi})$ to be admissible it must be self-financing. The self-financing condition is just

$$d\left(\hat{\varphi}_t N_t\right) \,=\, \varphi_s^1 \mathcal{M}_s d\left(\frac{P_s^1}{\mathcal{M}_s}\right) + \hat{\varphi}_s dN_s,$$

whence

$$\begin{aligned} N_t d\hat{\varphi}_t + dN_t d\hat{\varphi}_t \,&=\, \varphi_s^1 \mathcal{M}_s d\left(\frac{P_s^1}{\mathcal{M}_s}\right) = \varphi_t^1 \left\{(\eta_t^1 - \mu_t) P_t^1 dt + \sigma_t^1 P_t^1 dW_t\right\} \\ &=\, -\hat{\varphi}_t \frac{\hat{\sigma}_t}{\sigma_t^1 P_t^1} \left\{(\eta_t^1 - \mu_t) P_t^1 dt + \sigma_t^1 P_t^1 dW_t\right\}. \end{aligned}$$

Rearranging,

$$
\begin{aligned}
d\hat{\varphi}_t &= -\hat{\varphi}_t \frac{\hat{\sigma}_t}{\sigma_t^1 P_t^1 N_t} \left\{ (\eta_t^1 - \mu_t) P_t^1 dt + \sigma_t^1 P_t^1 dW_t \right\} \left\{ 1 - \frac{dN_t}{N_t} \right\} \\
&= \hat{\varphi}_t \frac{\hat{\sigma}_t}{N_t} \left\{ -\frac{\eta_t^1 - \mu_t}{\sigma_t^1} dt - dW_t \right\} \left\{ 1 - \frac{dN_t}{N_t} \right\} = \hat{\varphi}_t \frac{\hat{\sigma}_t}{N_t} \left\{ -J_t dt - dW_t + \frac{dW_t dN_t}{N_t} \right\} \\
&= \hat{\varphi}_t \frac{\hat{\sigma}_t}{N_t} \left\{ \left(\frac{\sigma_t^N}{N_t} - J_t \right) dt - dW_t \right\}.
\end{aligned}
$$

This is a Doléans exponential, which yields our solution:

$$
\hat{\varphi}_t = \hat{\varphi}_0 \exp\left(\int_0^t \left(\frac{\hat{\sigma}_s^2}{2N_s^2} - \frac{\hat{\sigma}_s}{N_s} J_s \right) ds - \frac{\hat{\sigma}_s}{N_s} dW_s \right).
$$

Noting that

$$
d \log N_t = \frac{dN_t}{N_t} - \frac{(dN_t)^2}{2N_t^2} = \frac{dF_t}{N_t} + \frac{\hat{\sigma}_t}{N_t} dW_t - \frac{\hat{\sigma}_t^2}{2N_t^2} dt,
$$

yields

$$
\hat{\varphi}_t = \hat{\varphi}_0 \exp\left(\int_0^t \frac{dF_s}{N_s} - d\log N_s - \frac{\hat{\sigma}_s}{N_s} J_s ds \right) = \frac{\hat{\varphi}_0}{N_t} \exp\left(\int_0^t \frac{dF_s}{N_s} - \frac{\hat{\sigma}_s}{N_s} J_s ds \right).
$$

Finally we can recover B_t explicitly. This is just G_t, the gain, and this is just our total wealth. As our wealth is all held in the numeraire, this is $\hat{\varphi}_t N_t$,

$$
B_t = \hat{\varphi}_t N_t = \hat{\varphi}_0 \exp\left(\int_0^t \frac{dF_s}{N_s} - \frac{\hat{\sigma}_s}{N_s} J_s ds \right).
$$

We are free to choose $\hat{\varphi}_0$ which completes the proof. □

11.4 The measure $\mathbb{B}$ is the risk-free measure

In [12], under the no arbitrage assumption of Corollary 10, Piterbarg identifies a measure $\mathbb{B} \sim \mathbb{P}$, given by

$$
\left. \frac{d\mathbb{B}}{d\mathbb{P}} \right|_{\mathcal{F}_t} := \exp\left(-\int_0^t J_u dW_u - \frac{1}{2} \int_0^t J_u^2 du \right),
$$

where, as above, $J_t := (\eta_t^i - \mu_t)/\sigma_t^i$, and here we assume that the Doléans exponential is a martingale.

Though in the case of a single Brownian driver the above measure change is fully specified by Piterbarg's assumptions on P^1, P^2 and μ there are many ways to complete the specification of the economy. In order to price derivatives we need to specify the economy fully and for the fully specified economy we can't avoid the existence of the finite variation process B. As we shall see below, modulo some technicalities, the measure $\mathbb{B}$ above is, in fact, precisely the risk-free measure corresponding to the (cash account) numeraire B.

Thus Piterbarg's goal of avoiding having to assume the existence of a risk-free rate by setting up an economy containing only collateralized assets is not achievable. Technically, once the economy has been properly specified, we end up in a similar (though not equivalent) situation to that if we had specified the economy via non-collateralized assets in terms of the role played by the finite variation process B. However there are good practical reasons for choosing collateralized assets as the underlyings in the market place and for an economy which includes collateralized assets as primitives we must revise the definition of an EMM. Before doing this we begin with a preliminary result.

Lemma 29. *Let $\mathcal{E}$ be an economy driven by a single Brownian motion W and comprising the collateralized trades with prices (P^1, P^2) and (at least one) other (non-dividend paying) assets. Further suppose that the condition of Corollary 27 holds, that N is a numeraire process for $\mathcal{E}$ and let B be defined as in Theorem 28.*
Then under $\mathbb{B}$ the processes $P_t^i/\mathcal{M}_t, i = 1, 2$, are local martingales and $\mathbb{B}$ is the unique measure equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ for which this holds. Furthermore B is the unique strictly positive finite variation process such that under $\mathbb{B}$ the process N/B is a local martingale.

Proof. By Girsanov's Theorem $W_\cdot - \int_0^\cdot J_s ds$ is a Brownian motion under $\mathbb{B}$ and so the drift for P^i under this measure is μP^i. It follows by Itô's formula that $P^i/\mathcal{M}$, $i = 1, 2$, are local martingales. Clearly $\mathbb{B}$ is the unique measure for which this holds.
From integration by parts it is easily checked that in order for N/B to be a local martingale under $\mathbb{B}$, where B is some finite variation process, we must have B satisfying the SDE

$$dB_t = B_t \frac{dN_t^{\text{fv}}}{N_t},$$

where N^{fv} denotes the finite variation part of N under $\mathbb{B}$. Note by Girsanov's Theorem that, under $\mathbb{B}$, $dN_t^{\text{fv}} = dF_t - \sigma_t^N J_t dt$ and thus the expression for B given in the last section is precisely the unique solution to this SDE. It follows that B is the unique finite variation process for which N/B is a local martingale under $\mathbb{B}$. Clearly B is strictly positive. □

The no arbitrage assumption of Corollary 27 is not sufficient to ensure that $(B, \mathbb{B})$ is a numeraire pair for the fully specified economy. The following result provides conditions to ensure that $\mathbb{B}$ can be interpreted as an EMM.

Lemma 30. *Let $\mathcal{E}$ be an economy as in Lemma 29 and containing the (non-dividend paying) assets A^i, $i = 1, \ldots, m$. Then, if for some strictly positive price process Π^i of the economy $\mathcal{E}$, the processes $M^{P^i:\Pi^i}\Pi^i/B, i = 1, 2$, and $A^j/B, j = 1, \ldots, m$, are martingales under $\mathbb{B}$ (rather than just local martingales), where*

$$M_t^{P^i:\Pi^i} := \int_0^t \frac{\mathcal{M}_s}{\Pi_s^i} d\left(\frac{P_s^i}{\mathcal{M}_s}\right)\left(1 - \frac{d\Pi_s^i}{\Pi_s^i}\right),$$

then $(B, \mathbb{B})$ is a numeraire pair for this economy. In particular, $\mathbb{B}$ is the unique EMM corresponding to numeraire B and the economy is arbitrage-free and complete.

Proof. As Π^i is a price process it follows by integration by parts that any measure for which $M^{P^i:\Pi^i}\Pi^i/B$ is a martingale will have $P^i/\mathcal{M}$ a local martingale and so, no matter how the economy is specified, the measure $\mathbb{B}$ will be the unique candidate EMM corresponding to numeraire B.
Now consider the standard economy $\tilde{\mathcal{E}} = (M^{P^1:\Pi^1}\Pi^1, M^{P^2:\Pi^2}\Pi^2, A^1, \ldots, A^m)$. Under the assumptions of the lemma and by standard theory this economy has a unique EMM $\mathbb{B}$ corresponding to numeraire B and thus is arbitrage-free and complete. Further we can price the collateralized derivatives having prices $P^i, i = 1, 2$.
Now suppose we replace the first two assets by our collateralized derivatives to form the economy $\mathcal{E}$ and consider a trading strategy in which we hold one unit of the collateralized derivative having price P^i and invest all the income from this derivative in the 'numeraire' Π^i. It is

straightforward to check that in order for the strategy to be self-financing we must hold $M^{P^i:\Pi^i}$ units of Π^i at any time and the resultant gain process is $M^{P^i:\Pi^i}\Pi^i$. Under the assumptions of the lemma this gain process rebased by the numeraire B will be a martingale under $\mathbb{B}$. Thus if we consider the set of admissible strategies for each of these economies to be those for which the numeraire-rebased gain process arising from a self-financing strategy is a martingale the two economies can be viewed as equivalent. $\square$

Remark 24: Note that, although here the economy is complete, we cannot necessarily price the corresponding uncollateralized derivatives that payoff P^i_T at some time T unless we have that P^i_T/B_T is integrable under $\mathbb{B}$. In practice we may want to set up our model so that this contraint is satisfied.

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